ON EQUIVARIANT SMOOTHING OF CUSP SINGULARITIES

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ABSTRACT. We generalize Looijenga's conjecture for smoothing surface cusp singularities to the equivariant setting. The result provides an evidence for the existence of the moduli stack of lci covers over semi-log-canonical surfaces.

CONTENTS

1. INTRODUCTION

A normal Gorenstein surface singularity $(\overline{V}, p)$ is called a *cusp* if the exceptional divisor of the minimal resolution is a cycle of smooth rational curves or a rational nodal curve, such that the exceptional divisor of the minimal resolution of $(\overline{V}, p)$ is an anti-canonical divisor of a rational surface. It is one type of minimally elliptic surface singularities in [15]. Let $\pi : V \to \overline{V}$ be the minimal resolution, and let

$$\pi^{-1}(p) = D = D_1 + \cdots + D_n \in |-K_V|.$$

The analytic germ $(\overline{V}, p)$ of the cusp singularity is uniquely determined by the self-intersections D_i^2 of D. When $n \geq 3$, we assume that $D_i \cdot D_{i\pm 1} = 1$. If $n = 2$, D is the union of two smooth rational curves that meet transversely at two distinct points. Since D is contractible, Artin's criterion for contractibility implies that the intersection matrix $[D_i \cdot D_j]$ is negative-definite.

Cusp singularities come naturally in dual pairs $(\overline{V}, p)$ and $(\overline{V}', p')$. Their resolution cycles D and D' under minimal resolutions are called dual cusp cycles. From [10], [9], there exists a non-algebraic, but compact complex analytic surface V whose only curves are the dual cycles D and D'. It is called the hyperbolic Inoue-Hirzebruch surface. Contracting these two cycles we get a singular surface $\overline{\overline{V}}$ with only two cusp singularities p, p'. We call p the dual cusp of the cusp p', and vice versa. We also call D the dual cycle of the cycle D', and vice versa.

Recall from [16], an anti-canonical pair (called a Looijenga pair) (Y, D) is a smooth rational surface Y, together with an anti-canonical divisor $D \in |-K_Y|$. The topology and geometry of Looijenga pairs were studied in [16], [6]. The smoothing of cusp singularities is close related to the geometry of Looijenga pairs. Looijenga studied the universal deformation of the hyperbolic Inoue-Hirzebruch surface V, and proposed the following conjecture, now a theorem:

Theorem 1.1. *(*[16, III Corollary 2.3], [8],[4, §5]*) The cusp singularity $(\overline{V}, p')$ admits a smoothing if and only if the dual cycle D of the dual cusp p is the anti-canonical divisor of a Looijenga pair (Y, D).*

Looijenga [16, III Corollary 2.3] proved that there exists a universal deformation of the hyperbolic Inoue-Hirzebruch surface V and gave a proof for the necessary condition of Theorem 1.1. The sufficient condition of Theorem 1.1 was first proven by Gross-Hacking-Keel [8] using mirror symmetry. For a Looijenga pair (Y, D), Gross-Hacking-Keel constructed the mirror family of (Y, D) using the log-Gromov-Witten invariants of (Y, D), and the mirror family gives the smoothing of the dual cusp p' of D. The natural existence of the dual cusps p, p' in the Inoue-Hirzebruch surface implies the mirror symmetry property. Later in [4, §5], Engel gave a proof of the sufficient condition of Theorem 1.1 using birational geometry–Type III degeneration of Looijenga pairs. The proof is beautiful and can be understood in a combinatorial way.

The main goal of this paper is to prove an equivariant version of Looijenga conjecture. Let us first discuss our motivation. For a cusp singularity germ $(\overline{W}, q)$, if the local embedded dimension is higher than 5, then [15, Theorem 3.13] showed that the singularity is not a locally complete intersection (l.c.i.) singularity. In [18, Proposition 4.1 (2)], Neumann and Wahl constructed a finite cover $(\overline{V}, p)$ of $\overline{W}$ with transformation finite group G so that $(\overline{V}, p)$ is a l.c.i. cusp (actually a hypersurface cusp). The finite cover is determined by the link Σ of the singularity germ $(\overline{W}, q)$, which is, by definition, the boundary of a neighborhood around the singularity p. The link Σ is a T^2-bundle over the circle S^1 and the first homology group $H_1(\Sigma, \mathbb{Z}) = \mathbb{Z} \oplus G'$, where G' is the torsion subgroup of $H_1(\Sigma, \mathbb{Z})$. The finite cover $\overline{V}$ is determined by the surjective morphism $H_1(\Sigma, \mathbb{Z}) = \mathbb{Z} \oplus G' \to G'$ up to automorphisms of $\pi_1(\Sigma)$. The transformation group G is obtained from the discriminant group G'. Thus we have a quotient map:

$$\mu : (\overline{V}, p) \to (\overline{W}, q) \tag{1.0.1}$$

such that $\overline{V}/G \cong \overline{W}$, and $\overline{V} \setminus \{p\} \to \overline{W} \setminus \{q\}$ is an unramified G-cover.

We are interested in the Gorenstein smoothings of $(\overline{W}, p)$ which are induced from G-equivariant smoothings of the cusp $(\overline{V}, q)$. For the cusp $(\overline{W}, q)$, we denote its dual cusp by q', and the corresponding singular Inoue-Hirzebruch surface by $(\overline{\overline{W}}, q, q')$, and the compact complex analytic Inoue-Hirzebruch surface by (W, E, E').

Since the two cusps $(\overline{\overline{W}}, q, q')$ are dual to each other, we present the smoothing of the cusp q' in the following theorem. Our main result is:

Theorem 1.2. *The cusp $(\overline{V}, p')$ admits a G-equivariant smoothing such that the quotient space induces a smoothing of the cusp $(\overline{W}, q')$ if and only if the dual cycle D of the cusp p lies as an anti-canonical divisor in a Looijenga pair (Y, D), and the pair (Y, D) admits a group G-action such that G acts freely on the complement $Y \setminus D$, and the quotient space $(Y, D)/G$, maybe after suitable resolution of singularities along D/G, gives a Looijenga pair (X, E) such that E is the dual cycle of the dual cusp q of q'.*

To prove the necessary condition of the theorem, we use the fact that for any finite subgroup G in the automorphism group of an Inoue-Hirzebruch surface $\overline{\overline{V}}$, the quotient space is still an Inoue-Hirzebruch surface $\overline{\overline{W}}$. We follow the method of [4] to prove the sufficient condition, by putting the finite group action into the combinatorial construction of Type III canonical degeneration pairs.

Our result Theorem 1.2 has applications for the moduli stack of lci covers defined in [12], where the author constructed the virtual fundamental class on the moduli space of lci covers over the semi-log-canonical surfaces. Surface cusp singularities are semi-log-canonical singularities in the construction of the moduli space of general type surface. The KSBA compactification of the moduli space $\overline{M}_{K^2,\chi}$ of general type surfaces, is the surface analogue of the moduli space of stable curves $\overline{M}_g$. Here $K^2 = K_S^2, \chi = \chi(\mathcal{O}_S)$ for a surface in $\overline{M}_{K^2,\chi}$. The boundary of $\overline{M}_{K^2,\chi}$ are given by singular surfaces with semi-log-canonical singularities; see [13] for more details. Let us only talk about the normal surface case. Then the log-canonical surface singularities, except the locally complete intersection singularities, quotient singularities, are given by simple elliptic singularities and cusp singularities. The cusp singularities with higher embedded dimension (> 5) are bad surface singularities, which have higher obstruction spaces (see [11]). These singularities cause trouble in the construction of a perfect obstruction theory on the moduli space $\overline{M}_{K^2,\chi}$ in [12].

In order to control such bad singularities, the author introduced the lci covering DM stack $\mathfrak{S}^{\mathrm{lci}} \to S$ for the semi-log-canonical surface S with higher embedded dimension cusp singularities. The stacky structure around a cusp singularity $p \in S$ is given by the DM stack $[\widetilde{S}/G]$, where $(\widetilde{S}, p) \to (S, q)$ is the lci cover as in [18, Proposition 4.1] and G is the transformation group. Suppose that there is a $\mathbb{Q}$-Gorenstein flat family $\mathcal{S} \to T$ of semi-log-canonical surfaces, and if there are cusp singularities with higher embedded dimension on the fiber surface $\mathcal{S}_t$, then we can take the lci cover to get the lci covering DM stack. But it is hard to see of the lci covering DM stacks form a flat family. But if there is a lci covering DM stack $\mathfrak{S}^{\mathrm{lci}} \to S$ to a semi-log-canonical surface S with only cusp singularities except the l.c.i. singularities, then any G-equivariant smoothing of $\widetilde{S}$ locally gives a smoothing of the lci covering DM stack $\mathfrak{S}^{\mathrm{lci}} \to S$. Thus it gives a smoothing of the underlying semi-log-canonical surface S. In [12, §4.3.6], the author construct the moduli stack $\overline{M}^{\mathrm{lci}}_{K^2,\chi}$ of lci covers over the component $\overline{M}_{K^2,\chi}$, based on the assumption that every $\mathbb{Q}$-Gorenstein family of semi-log-canonical surfaces in $\overline{M}_{K^2,\chi}$ can be obtained from a family of lci covering DM stacks. Thus, from Theorem 1.2 we have that:

Corollary 1.3. *The moduli space of* lci *covers over s.l.c. surfaces defined in* [12, §5.3.6] *exists if there are bad cusp singularities (with embedded dimension* > 5*) on the semi-log-canonical surfaces S in the moduli space $\overline{M}_{K^2,\chi}$ such that the $\mathbb{Q}$-Gorenstein smoothing of these cusp singularities are induced by the G-equivariant smoothing of their* lci *covers.*

1.1. **Related work.** There are two classes of log canonical surface germ singularities (S, p). The index of the singularity p is, by definition, the least integer N such that $\omega_S^{[N]} := (\omega_S^{\otimes N})^{**} = \mathcal{O}_S(N \cdot K_S)$ is invertible, where K_S is the canonical class of S. In the case of $N = 1$, the germ singularities (S, p) are given by simple elliptic singularities and cusp singularities. Our study above focuses on the index one case, and a key point of Theorem 1.2 is that there exist finite group G-actions on the Inoue-Hirzebruch surface $(\overline{\overline{V}}, p, p')$ (only two points p, p' are fixed by G) such that the quotient $(\overline{\overline{V}}, p, p')/G$ is still an Inoue-Hirzebruch surface $(\overline{\overline{W}}, q, q')$. Also in this setting we have to work on analytic complex surfaces.

If the index $N > 1$, then from [13, Theorem 4.24], the germ singularities (S, p) are given by $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$ quotients of simple elliptic singularities, and $\mathbb{Z}_2$ quotients of cusps, and degenerate cusps. Degenerate cusp singularities are non-normal surface singularities. The $\mathbb{Z}_2$ quotient-cusp singularities are rational singularities, which can not be cusp singularities any more. For such germ singularities, the link Σ is a rational homology sphere. Let $G = H_1(\Sigma, \mathbb{Z})$ be the finite abelian group. In [18], Neumann-Wahl constructed a universal abelian cover $(\widetilde{S}, q) \to (S, p)$ which is a Galois G-cover. The germ $(\widetilde{S}, q)$ is a locally complete intersection cusp singularity. The local defining equations of this locally complete intersection cusp singularity $(\widetilde{S}, q)$ were given in [18, Theorem 5.1]. Since locally complete intersection singularity admits a G-equivariant smoothig, its quotient gives the smoothing of the singularity (S, p). This also shows that our moduli stack of lci covers over s.l.c. surfaces defined in [12, §5.3.6] exists if there are such quotient-cusp singularities.

On the other hand, in [22], A. Simonetti studied the Looijenga conjecture for $\mathbb{Z}_2$-equivariant smoothings of cusps singularities using log Gromov-Witten theory and the techniques in [8]. Thus it is interesting to study the G-equivariant smoothing of $(\widetilde{S}, q)$ which induces smoothings of the $\mathbb{Z}_2$ quotient-cusp (S, p). Since the quotient $(\widetilde{S}, q)/G = (S, p)$ is not cusp singularity anymore, this provides difficulties for the construction of the corresponding Looijenga pairs.

Acknowledgments. Y. J. thanks Chen Jiang, Yuchen Liu for valuable discussion on semi-log-canonical singularities. Y. J. thanks P. Engel for his correspondence of the proof of Looijenga conjecture and the wonderful talk at Kansas. This work is partially supported by Simon Collaboration Grant.

2. Inoue-Hirzebruch surfaces with a finite group action

We study the Inoue-Hirzebruch surfaces together with a finite group action.

2.1. **Inoue-Hirzebruch surfaces.** Let us recall the construction of Inoue-Hirzebruch surface V in [10], [9], [16, III, §2.], [8].

Let $M = \mathbb{Z}^2$ be a rank two lattice, and $\sigma \in SL_2(\mathbb{Z})$ be a hyperbolic matrix. Then σ has two real eigenvalues $\lambda, \frac{1}{\lambda}$ for $\lambda > 1$. Let v_1, v_2 be the two eigenvectors corresponding to $\lambda_1 = \frac{1}{\lambda}, \lambda_2 = \lambda$ so that $v_1 \wedge v_2 > 0$. Let $\overline{C}, \overline{C}'$ be two strictly convex cones spanned by v_1, v_2 and $v_2, -v_1$. Let C, C' be their interiors which are both preserved by σ. Denote by

$$\mathfrak{D} := \{z = x + iy \in \mathbb{C}^2/M | y \in \mathbb{H}\}$$

where $\mathbb{H}$ is the upper half-plane. Then the finite cyclic group generated by σ acts freely and properly discontinuously on $\mathfrak{D}$. The quotient surface $\overline{\overline{V}}' := \mathfrak{D}/\langle\sigma\rangle$ is the open Inoue-Hirzebruch surface. The compactification (Inoue-Hirzebruch surface) $\overline{\overline{V}} := \overline{\overline{V}}' \cup \{p, p'\}$ is obtained by adding two singular cusp points p, p'.

Let

$$U'_C := \{z = x + iy \in \mathbb{C}^2/M | y \in C\}; \quad U'_{C'} := \{z = x + iy \in \mathbb{C}^2/M | y \in C'\}$$

Then we have two neighborhoods $\overline{\overline{V}}'_C := U'_C/\langle\sigma\rangle$ and $\overline{\overline{V}}'_{C'} := U'_{C'}/\langle\sigma\rangle$ in $\overline{\overline{V}}'$. Let

$$\overline{\overline{V}}_C := \overline{\overline{V}}'_C \cup \{p\}; \quad \overline{\overline{V}}_{C'} := \overline{\overline{V}}'_{C'} \cup \{p'\}.$$

Then $(\overline{\overline{V}}_C, p)$ and $(\overline{\overline{V}}_{C'}, p')$ are the two singularity germs for the cusps p and p', respectively in $\overline{\overline{V}}$.

Taking the resolutions of singularities for p, p', we get a smooth compact complex surface V with two cycles of rational curves

$$D = D_0 + D_1 + \cdots + D_n; \quad D' = D'_0 + D'_1 + \cdots + D'_s$$

corresponding to p, p' respectively. Let $V := \overline{\overline{V}}' \cup \{D, D'\}$. Then we call V the Inoue-Hirzebruch surface with only cycles of curves D and D'. Let

$$(d_0, d_1, \cdots, d_n); \quad (d'_0, d'_1, \cdots, d'_s)$$

be the cycle of D and D' given by negative self-intersection numbers, where

$$d_i = \begin{cases} -D_i^2 & \text{if } n > 0 \\ 2 - D_i^2 & \text{if } n = 0. \end{cases}$$

The numbers d'_i is defined similarly. Since both D and D' are contractible, the intersection matrix $[D_i \cdot D_j]$ is negative definite, which implies that $d_i \geq 2$ for all i, and $d_i \geq 3$ for some i. The generator σ is given by:

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & d_0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & d_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & d_n \end{pmatrix}$$

The duality property of the dual cusps D and D' implies that the cycles $(d_0, \cdots, d_n)$ and $(d'_0, \cdots, d'_s)$ have the following properties: if

$$(d_0, \cdots, d_n) = (a_0 + 3, \underbrace{2, \cdots, 2}_{b_0}, \cdots, a_k + 3, \underbrace{2, \cdots, 2}_{b_k}) \tag{2.1.1}$$

where $a_i, b_i \geq 0$. Then the negative self-intersections $(d'_0, \cdots, d'_s)$ is obtained from D by:

$$(d'_0, \cdots, d'_s) = (b_0 + 3, \underbrace{2, \cdots, 2}_{a_0}, \cdots, b_k + 3, \underbrace{2, \cdots, 2}_{a_k}). \tag{2.1.2}$$

2.2. **Finite group action on Inoue-Hirzebruch surfaces.** Let $\mathrm{Aut}(\overline{\overline{V}})$ be the automorphism group of the Inoue-Hirzebruch surface $\overline{\overline{V}}$. Pinkham [20], and Prokhorov-Shramov [21] studied the automorphism group $\mathrm{Aut}(\overline{\overline{V}})$. We first consider the automorphism group $\mathrm{Aut}(\overline{\overline{V}}')$ for the open Inoue-Hirzebruch surface. Let $G \subset \mathrm{Aut}(\overline{\overline{V}}')$ be a finite order subgroup, then from the topology of $\overline{\overline{V}}'$ we have:

Proposition 2.1. *([21, Lemma 7.1]) The action of G on $\overline{\overline{V}}'$ is free and the quotient $\overline{\overline{W}}' = \overline{\overline{V}}'/G$ is still an Inoue-Hirzebruch surface.*

Thus the finite group G action on $\overline{\overline{V}}'$ can be naturally extended to $\overline{\overline{V}} = \overline{\overline{V}}' \cup \{p, p'\}$ with the two dual cusps p, p' fixed. We also extend the G-action to the compact Inoue-Hirzebruch surface $V = \overline{\overline{V}}' \cup \{D, D'\}$ following [20, §2].

From [20, Page 302], for each $k \in \mathbb{Z}$, we take a $\mathbb{C}^2$ given by coordinates (u_k, v_k). We glue all the infinite $\mathbb{C}^2$'s (indexed by $k \in \mathbb{Z}$) by:

$$\begin{cases} u_{k+1} = u_k^{d_k} v_k; \\ v_{k+1} = \frac{1}{u_k}. \end{cases} \tag{2.2.1}$$

Let A denote such a space. Note that $D_k \cong \mathbb{P}^1$ is given by $\{u_{k+1} = v_k = 0\}$ and $D_k^2 = -d_k$. The group $\langle \sigma \rangle$ acts on A freely by:

$$\sigma(u_k, v_k) = (u_{k+n}, v_{k+n}).$$

Then we have an isomorphism:

$$\Phi : A - \bigcup_{k \in \mathbb{Z}} D_k \cong U'_C$$

given by:

$$\begin{cases} 2\pi i z_1 = \omega \log u_0 + \log v_0; \\ 2\pi i z_2 = \omega' \log u_0 + \log v_0 \end{cases} \tag{2.2.2}$$

where $\omega = \overline{[d_0, \cdots, d_n]}$ is the irrational number which has a purely period modified fraction expansion, and ω' is its conjugate. (Here we also can identify $\mathbb{Z}^2$ (as a $\mathbb{Z}$-module) generated by $1, \omega$). Recall that $U'_C := \{z = x + iy \in \mathbb{C}^2/M | y \in C\}$. We consider $\Phi^{-1}(\mathbb{H} \times \mathbb{H}/M)$ and Φ is compatible with the action of $\langle \sigma \rangle$ on (2.2.2). Then we glue $A/\langle \sigma \rangle$ to $\mathbb{H} \times \mathbb{H}/M \rtimes \langle \sigma \rangle \subset U'_C/\langle \sigma \rangle = \overline{\overline{V}}'_C$ and we get a neighborhood of D in V. We denote this neighborhood by V_C.

From [20, §Step II], the group G acts on the space V_C. We explain the detail action. First our lattice is $M = \mathbb{Z}^2$, and the Inoue-Hirzebruch surface is

$$\overline{\overline{V}}' = \mathfrak{D}/\langle \sigma \rangle = (\mathbb{H} \times \mathbb{C})/M \rtimes \langle \sigma \rangle,$$

see §2.1. From [10], the Inoue-Hirzebruch surface $\overline{\overline{V}}' \cong \mathbb{R} \times \tau$, where τ is a $S^1 \times S^1$-bundle pver S^1, and
$$H_1(\tau, \mathbb{Z})_{\mathrm{tor}} = M/(\sigma - 1)M.$$
Here τ is the link of the singularity p, or p'. Now let
$$\overline{M} := (\sigma - 1)^{-1}M$$
be a new $\mathbb{Z}$-module such that $\overline{M} \cong M$ since $(\sigma - 1)\overline{M} = M$. Let $U_M \subset SL_2(\mathbb{Z})$ be the group of hyperbolic matrices. Then from [20, §2, Theorem], the full complex automorphism group of $V, \overline{\overline{V}}$ is given by
$$G(\overline{M}, U_M)/G(M, \langle \sigma \rangle)$$
such that our finite group $G \subset G(\overline{M}, U_M)/G(M, \langle \sigma \rangle)$. Here $G(\overline{M}, U_M) = \overline{M} \rtimes U_M$ and $G(M, \langle \sigma \rangle) = M \rtimes \langle \sigma \rangle$.

Now the part $U_M/\langle \sigma \rangle$ of the automorphism group naturally extends to $V, \overline{\overline{V}}$. For the part $\overline{M}/M$, for any element $m \in \overline{M}/M$ representing an element in G, we have m acts on $\mathbb{H} \times \mathbb{H}$ by
$$m \cdot (z_1, z_2) = (z_1 + m, z_2 + m'),$$
where m' is the conjugate of m. By equation (2.2.2), m acts on the 0-th coordinate chart of A by
$$m \cdot (u_0, v_0) = (e^{2\pi i m_1} u_0, e^{2\pi i m_2} v_0)$$
where $m = m_1\omega + m_2$, $m_1, m_2 \in \mathbb{Q}$, and $\omega = \overline{[d_0, \cdots, d_n]}$, $M \cong M(\omega)$ which is generated by $1, \omega$. By iterating equation (2.2.1), we get
$$\begin{cases} u_r = u_0^{p_r} v_0^{q_r}; \\ v_r = u_0^{-p_{r-1}} v_0^{-q_{r-1}} \end{cases}$$
where
$$\begin{pmatrix} p_r & q_r \\ -p_{r-1} & -q_{r-1} \end{pmatrix} = \begin{pmatrix} d_{r-1} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} d_{r-2} & 1 \\ -1 & 0 \end{pmatrix} \cdots \begin{pmatrix} d_0 & 1 \\ -1 & 0 \end{pmatrix}.$$
Set
$$N := \begin{pmatrix} p_r & q_r \\ -p_{r-1} & -q_{r-1} \end{pmatrix}.$$
By [20, Step II], $(\omega, 1)N = \sigma(\omega, 1)$. So m acts on the r-th coordinate chart of A by
$$m \cdot (u_r, v_r) = (e^{2\pi i (p_r m_1 + q_r m_2)} u_r, e^{2\pi i (-p_{r-1} m_1 - q_{r-1} m_2)} v_r).$$
Then the $\overline{M}$-action decends to the quotient $\overline{M}/M$ on $A/\langle \sigma \rangle$. Thus, $G \subset G(\overline{M}, U_M)/G(M, \langle \sigma \rangle)$ acts on V_C. Although the quotient V_C/G maybe not exactly the resolution of a cusp singularity, but a resolution of the quotient singularities will give a cusp.

Example 1. *Consider the hypersurface cusp $\{x^3 + y^3 + z^5 + xyz = 0\}$ whose resolution cycle is given by $(2, 5)$. The cyclic group $\mu_6 = \langle \zeta \rangle$ acts on the cusp by:*
$$x \mapsto \zeta x; \quad y \mapsto \zeta^5 y; \quad z \mapsto \zeta^3 z.$$
Then from [20, Example] *the quotient is given by the hypersurface cusp $\{x^2 + y^4 + z^7 + xyz = 0\}$ and the resolution cycle of this cusp is $(2, 2, 4)$.*

The quotient by the subgroup $\mu_3 = \langle \zeta^2 \rangle$ gives the cusp with resolution cycle $(3, 2, 2, 2, 2, 2)$, and the quotient by the subgroup $\mu_2 = \langle \zeta^3 \rangle$ gives the cusp with resolution cycle (8).

3. LOOIJENGA PAIR AND TYPE III CANONICAL DEGENERATION

3.1. **Looijenga pairs.** We introduce a finite group action on a Looijenga pair.

Definition 3.1. *A Looijenga pair (Y, D) is given by a smooth rational surface Y, together with a connected singular nodal curve $D \in |-K_Y|$.*

Since the arithmetic genus $p_a(D) = 1$, D is either an irreducible rational nodal curve, or a cycle of smooth rational curves. For such a D, we have $H_1(D, \mathbb{Z}) = \mathbb{Z}$. Thus fixing a. generator of $H_1(D, \mathbb{Z})$ gives an orientation on D, and we label D as $D = D_1 + \cdots + D_n$. The length of D is $\ell(D) = n$. We call D negative-definite, if the intersection matrix $[D_i \cdot D_j]$ is negative-definite.

Definition 3.2. *The charge of a Looijenga pair $Q(Y, D)$ is defined by the formula:*

$$Q(Y, D) := 12 + \sum_{i=1}^{n}(d_i - 3) = \sum_{i=1}^{n}(a_i - b_i).$$

Let D' be the dual cycle of D, then $Q(Y, D')$ is given by interchanging the a_i with b_i from (2.1.1) and (2.1.2). We have $Q(Y, D) + Q(Y, D') = 24$. A Looijenga pair (Y, D) is called a toric pair if Y is a toric variety, and D is its boundary divisor.

There are good properties of the charge $Q(Y, D)$ for a Looijenga pair (Y, D), which is related to the geometry and topology property of the pair. We only mention some useful ones, more properties can be found in [6], [7, §4]. If D is negative-definite, then $Q(Y, D) \geq 3$. Also

Proposition 3.3. ([6, Lemma 1.2]) *For a Looijenga pair (Y, D), the charge*

$$Q(Y, D) = \chi_{\text{top}}(Y - D)$$

The charge of the Looijenga pair measures how far the pair is being a toric pair. The above result implies that

Proposition 3.4. ([6, Lemma 2.7]) *If (Y, D) is an anticanonical pair, then $Q(Y, D) \geq 0$ and (Y, D) is toric if and only if $Q(Y, D) = 0$.*

From [7, Definition (4.4)], suppose that D represents a cusp, meaning that it is the resolution cycle of a cusp singularity, then we say that D is rational if there exists a Looijenga pair (Y, D) so that D is the anti-canonical divisor of the rational surface Y. If D is a cusp such that the dual D' to D is rational, we say that D has a rational dual.

Proposition 3.5. ([7, Theorem (4.5)]) *If D has a rational dual, then the charge $Q(D) \leq$* 21.

Conversely, if D is a cusp and $\ell(D) \leq 3$, $Q(D) \leq 21$, then D has a rational dual except in the following cases: $(4, 11), (7, 8), (2, 4, 12), (2, 8, 8), (3, 3, 12), (3, 4, 11), (3, 7, 8), (4, 4, 10), (4, 6, 8), (4, 7, 7), (5, 5, 8)$.

3.2. **Corner blow-ups and internal blow-ups.** We introduce corner blow-ups and internal blow-ups for the Looijenga pairs. For a Looijenga pair (Y, D), suppose that there is an exceptional curve E meaning $E \cong \mathbb{P}^1$, $E^2 = -1$, then contracting E gives a Looijenga pair

$$\pi : (Y, D) \to (\overline{Y}, \overline{D}).$$

If $E \subset D$ is a component, then E contracts to a node of the cycle $\overline{D}$. We call this type blow-up the corner blow-up.

If $E \not\subset D$, then E meets with D at a smooth point of D. Then in this case E contracts to a smooth point of the cycle $\overline{D}$. We call this type blow-up the internal blow-up.

Proposition 3.6. *(*[6, Lemma 2.2]*) If there is a Looijenga pair (Y, D) such that the negative self-intersection sequence is $(d_1, \cdots, d_n)$ and the charge is $Q(Y, D)$, then we have*

(1) Let $\widetilde{Y} \to Y$ be an internal blow-up at the point $p \in D_i^\circ$ (the interior part of D_i), then $\ell(\widetilde{D}) = \ell(D)$, and, under the natural labeling of $\widetilde{D}$, the negative self-intersection sequence of $(\widetilde{Y}, \widetilde{D})$ is $(d_1, \cdots, d_{i-1}, d_i + 1, d_{i+1}, \cdots, d_n)$.

(2) If $\widetilde{Y} \to Y$ is a corner blow-up of Y at the point $p \in D_i \cap D_{i+1}$, then $\ell(\widetilde{D}) = \ell(D) + 1$. If $\ell(D) = 1$, i.e., D is irreducible, then the negative self-intersection sequence of $(\widetilde{Y}, \widetilde{D})$ is $(d_1 + 4, 1)$. If $\ell(D) \geq 2$, and for an appropriate labeling of the components of $\widetilde{D}$, the negative self-intersection sequence of $(\widetilde{Y}, \widetilde{D})$ is

$$(d_1, \cdots, d_{i-1} + 1, 1, d_{i+1} + 1, \cdots, d_n).$$

(3) If $\widetilde{Y} \to Y$ is an internal blow-up of Y, then $Q(\widetilde{Y}, \widetilde{D}) = Q(Y, D) + 1$; and, if $\widetilde{Y} \to Y$ is a corner blow-up of Y, then $Q(\widetilde{Y}, \widetilde{D}) = Q(Y, D)$.

From [8, Proposition 1.3], if we have a Looijenga pair (Y, D), there exists a sequence of corner blow-ups (Y', D') such that (Y', D') has a toric model. This means (Y', D') can be obtained from a toric Looijenga pair $(\overline{Y}, \overline{D})$ by internal blow-ups at some number of smooth points.

3.3. **Finite group action on Looijenga pairs.**

Definition 3.7. *Let (Y, D) and (Y', D') be two Looijenga pairs. An isomorphism between these two Looijenga pairs is given by an isomorphism*

$$f : Y \to Y'$$

such that $f(D_i) = D_i'$ for $i = 1, 2, \cdots, n$, and f is compatible with the orientation of D and D'. We let $\mathrm{Aut}(Y, D)$ be the automorphism groups for (Y, D).

Let G be a finite group. We say a Looijenga pair (Y, D) admits a G-action if $G \subset \mathrm{Aut}(Y, D)$ is a finite subgroup of the automorphism group.

We consider a special finite group G-action on a Looijenga pair. Let us now restrict to negative-definite Looijenga pairs (Y, D). Artin's criterion for contractibility implies that D can be contracted to a singular cusp point

$$\pi : (Y, D) \to (\overline{Y}, p).$$

Definition 3.8. *A finite group G-action on a negative-definite Looijenga pair is called* hyperbolic type *if the G-action is free on $Y - D$, and there exists a open neighborhood $V_D \subset Y$ if D such that V_D is isomorphic to the neighborhood V_C constructed in §2.2* *and the G-action on V_D is induced by the action on V_C. Moreover, the quotient space $(V_D - D)/G$ is isomorphic to another open analytic space $V_{C'}$ in §2.2, so that adding one cusp point q to $(V_D - D)/G$ we get a neighborhood $\overline{\overline{V}}_{C'}$ of q.*

Remark 3.9. *Definition 3.8 implies that if a G-action on (Y, D) is hyperbolic, then the quotient $(Y/D)/G = (X, E)$ is also a Looijenga pair, where E is slao a cusp.*

Remark 3.10. *In general it is interesting to study the symplectic finite group G action on a pair (Y, ω), where Y is a rational surface, and ω is a symplectic form. One can ask a question when the quotient $(Y, \omega)/G$ still a rational surface.*

3.4. **The toric model of Looijenga pairs with a finite group action.** We generalize the process of internal blow-ups and corner blow-ups in the equivariant setting using the main result in [1, Theorem 0.1].

Proposition 3.11. *Let (Y, D) be a negative definite Looijenga pair endowed with a finite group G-action. Suppose that the action is hyperbolic. Then we can extend the G-action to the corner blow-ups and internal blow-ups, such that the toric model $(Y^{\text{toric}}, D^{\text{toric}})$ also admits an action of G with quotient the toric model of the quotient $(Y, D)/G = (X, E)$.*

Proof. The action of the finite group G on the Looijenga pair (Y, D) is hyperbolic, which means G acts freely on $Y - D$. The pair (Y, D) is negative definite, so the negative self-intersection numbers $d_i \geq 2$ for any i, and some $d_j \geq 3$. Therefore, there are no -1-curves on Y which are fixed by G. The pair (Y, D) is already G-minimal.

Since there is a neighborhood $V_D \subset Y$ such that G preserves V_D, then the finite group G lies in $\mathrm{Aut}_{\mathbb{C}}(D \subset Y)$. Thus, [1, Theorem 0.1], the internal blow-ups and corner blow-ups can be made into G-equivariant. Therefore, we have the following diagram:

$$
\begin{tikzcd}
(Y^{\text{toric}}, D^{\text{toric}}) \arrow[d, "\pi"'] & (\widetilde{Y}, \widetilde{D}) \arrow[l, "\text{internal blow-ups}"'] \arrow[r, "\text{corner blow-ups}"] \arrow[d, "\pi"'] & (Y, D) \arrow[d, "\pi"'] \\
(X^{\text{toric}}, E^{\text{toric}}) & (\widetilde{X}, \widetilde{E}) \arrow[l, "\text{internal blow-ups}"'] \arrow[r, "\text{corner blow-ups}"] & (X, E)
\end{tikzcd}
$$

□

The quotient $(Y^{\text{toric}}, D^{\text{toric}})/G$ must give the birational model of (X, E), since it can be obtained from $(Y^{\text{toric}}, D^{\text{toric}})/G$ by internal blow-ups and corner blow-ups.

3.5. **Universal deformation of Inoue-Hirzebruch surfaces.** Let $(\overline{\overline{V}}, p, p')$ be the Inoue-Hirzebruch surface with two dual cusp singularities. Looijenga [16, III Corollary 2.3] proves that the surface $\overline{\overline{V}}$ admits a universal deformation. Suppose that there is a finite group G action on the Inoue-Hirzebruch surface $\overline{\overline{V}}$. The proof in [16, II §2] works in the G-equivariant case, therefore implies that $\overline{\overline{V}}$ admits a universal G-deformation.

Let
$$\overline{\overline{\mathcal{V}}} \to \Delta$$
be a G-equivariant deformation of $(\overline{\overline{V}}, p, p')$ along the cusp p'. So $\overline{\overline{\mathcal{V}}}_0 = \overline{\overline{V}}$, and the cusp p keeps constant. Any fiber $\overline{\overline{\mathcal{V}}}_t (t \neq 0)$ is a surface with a single cusp singularity $p = p_t$.

We resolve p_t in the family under the group G-action and get a family

$$\pi : \mathcal{Y} \to \Delta$$

such that $\mathcal{Y}_0 = \overline{V}_0$ and $(\overline{V}_0, p')$ is the partially contracted Inoue-Hirzebruch surface from (V, D, D') with only cusp singularity p'. For $t \neq 0$, $\mathcal{Y}_t$ is a simply connected surface with an anti-canonical divisor $D \in |-K_Y|$. Thus $\mathcal{Y}_t$ is rational. Since we do a G-equivariant simultaneous resolution of the singularities p_t of the family $\overline{\overline{\mathcal{V}}} \to \Delta$, for each fiber $\mathcal{Y}_t$ in the resolution family $\mathcal{Y} \to \Delta$, there must exist a subgroup $H \subset G$ acting faithfully on the fiber $\mathcal{Y}_t$. Our group G acts originally on the surface $\overline{\overline{V}}$ and $\overline{\overline{\mathcal{V}}}$. After taking the G-equivariant resolution, the group H acts on the fiber $\mathcal{Y}_t$ which preserves the anti-canonical divisor $D_t \in |-K_{\mathcal{Y}_t}|$.

It is ready to introduce the Type III degeneration pairs. Consider the G-equivariant family

$$\pi : \mathcal{Y} \to \Delta.$$

We first make the construction of Friedman-Miranda [7] for the Type III canonical degeneration pair work in the G-equivariant setting. We construct the following Type III degeneration pairs.

$$\mathfrak{X}_0 = \bigcup_{i=0}^{n} V_i \tag{3.5.1}$$

where

(1) $V_0 = (V, D, D')$, the compact Inoue-Hirzebruch surface. For $i > 0$, the normalization $\widetilde{V}_i$ of V_i is a smooth rational G-surface.
(2) We let D_{ij} be the irreducible double curve of $\mathfrak{X}_0$ lying on V_i and V_j (in the case V_i is not normal, we may have $i = j$). Let $D_i = \cup D_{ij} \subset V_i$ and $\widetilde{D}_i = \pi^{-1}(D_i)$ under $\pi : \widetilde{V}_i \to V_i$. Then $(\widetilde{V}_i, \widetilde{D}_i)$ is a G-Looijenga pair. For $i = 0$, $D_0 = D'$.
(3) (Triple point formula) For the double curve D_{ij} above,

$$(D_{ij}|_{\widetilde{V}_i})^2 + (D_{ij}|_{\widetilde{V}_j})^2 = \begin{cases} -2, & D_{ij} \text{ are smooth;} \\ 0, & D_{ij} \text{ are nodal.} \end{cases}$$

(4) The dual complex of the central fiber $\mathfrak{X}_0$ is a triangulation of sphere.
(5) The quotient V_0/G is, after suitable resolution of singularities, another Inoue-Hirzebruch surface (W, E, E').

Theorem 3.12. *Suppose that the invariant part $(\Omega_{\mathfrak{X}_0})^G$ of the cotangent sheaf $\Omega_{\mathfrak{X}_0}$ is nonzero. Then we have a G-equivariant family $\mathfrak{X} \to \Delta$ such that $\mathcal{D} \in |-K_{\mathfrak{X}}|$ and $\mathcal{D}_t = D_t \in \mathcal{Y}_t$ and $\mathfrak{X}_0$ is the variety in (3.5.1).*

Proof. We prove the theorem by first generating [7, Lemma 2.9]. Let $T^0_{\mathfrak{X}_0}$ and $T^1_{\mathfrak{X}_0}$ be the tangent sheaves so that

$$T^i_{\mathfrak{X}_0} = \mathcal{E}xt^i(\Omega^1_{\mathfrak{X}_0}, \mathcal{O}_{\mathfrak{X}_0}).$$

The the global tangent spaces are defined by

$$\mathbb{T}^i_{\mathfrak{X}_0} = \mathrm{Ext}^i(\Omega^1_{\mathfrak{X}_0}, \mathcal{O}_{\mathfrak{X}_0}).$$

Recall that the variety $\mathfrak{X}_0$ is called d-semi-stable, if $T^1_{\mathfrak{X}_0} = \mathcal{O}_Q$, where $Q \subset \mathfrak{X}_0$ is the singular locus. We have that

Lemma 3.13. *For the variety $\mathfrak{X}_0$, there always exists an $\mathfrak{X}'_0$, with the same Inoue-Hirzebruch component and double curve D_0 such that $\mathfrak{X}'_0$ is d-semi-stable.*

Proof. We generalize [5, Proposition (5.14)] in this setting. Recall that D_{ij} is the double curve in V_i and V_j. $D_i = \cup D_{ij}$ and we set $E := \cup D_i$. We let $D^0_{ij} := D_{ij} - T$, where T is the triple point locus. We know that D_{ij} is smooth (D_{ij} is not a nodal rational curve since $n > 1$). We show that there exists a choice of isomorphisms

$$\varphi_{ij} : D^0_{ij} \subset V_i \xrightarrow{\sim} D^0_{ij} \subset V_j$$

where the extension $\overline{\varphi}_{ij}$ of φ_{ij} to D_{ij} fixes the triple points and the surface $\mathfrak{X}_0$ is d-semi-stable by the gluing of $\overline{\varphi}_{ij}$. The triple point formula implies that $(D_{ij}|_{\widetilde{V}_i})^2$ or $(D_{ij}|_{\widetilde{V}_j})^2$ is nonzero. Our finite group G acts on $\mathfrak{X}_0$, and the surface $\mathfrak{X}_0 = \cup_{i=0}^n V_i$, where $V_0 = (V, D, D')$ is an Inoue-Hirzebruch surface with a G-action. For $i > 0$, each $\widetilde{V}_i \to V_i$ is a rational G-surface. We follow the same proof as in [7, (5.14)]. Let

$$G_{ij} = \{\text{divisors of degree zero on } D^0_{ij}\} / \operatorname{div}(f)$$

where the f are functions on D_{ij}, which are not zero and ∞ at the triple points t_1, t_2, and $f(t_1) = f(t_2)$. Then we have that

$$G_{ij} \cong \mathbb{C}^* \subset \operatorname{Pic}^0(D_{ij}) \subset \operatorname{Pic}^0(E).$$

Let $\widetilde{E} \to E$ be the normalization and consider the following exact sequence

$$0 \to H^0(\mathcal{O}^*_E) \to H^0(\mathcal{O}^*_{\widetilde{E}}) \to H^0(\mathcal{O}^*_{\widetilde{E}}/\mathcal{O}^*_E) \to H^1(\mathcal{O}^*_E) \to 0, \tag{3.5.2}$$

the $\operatorname{Pic}(E)$ is determined by the gluing from $H^0(\mathcal{O}^*_{\widetilde{E}}/\mathcal{O}^*_E)$. From [5, Definition 1.9], we have

$$\mathcal{O}_{D_i}(-\mathfrak{X}_0) = (I_{D_i}/I^2_{D_i}) \otimes_{\mathcal{O}_{D_i}} (I_{V_i}/I_{V_i}J_{D_i})$$

and

$$\mathcal{O}_E(-\mathfrak{X}_0) = (I_{V_0}/I_{V_0}I_E) \otimes_{\mathcal{O}_E} (I_{V_1}/I_{V_1}I_E) \otimes_{\mathcal{O}_E} \cdots \otimes_{\mathcal{O}_E} (I_{V_n}/I_{V_n}I_E)$$

where

$$\begin{cases} I_{D_i} = \text{ideal sheaf of } D_i \text{ in } V_i; \\ I_{V_i} = \text{ideal sheaf of } V_i \text{ in } \mathfrak{X}_0; \\ J_{D_i} = \text{ideal sheaf of } D_i \text{ in } \mathfrak{X}_0. \end{cases}$$

From [5, Definition 1.13], $\mathfrak{X}_0$ is d-semi-stable if $\mathcal{O}_E(\mathfrak{X}_0) = \mathcal{O}_E$, which is equivalent to $T^1_{\mathfrak{X}_0} = \mathcal{O}_Q$. The locally free sheaf $\mathcal{O}_E(-\mathfrak{X}_0)$ is defined by the trivial bundles $\mathcal{O}_{D_{ij}}$, plus the gluing defined by using

$$z_i z_j z_k \in H^0(\mathcal{O}_{D_{ij}}(-V_i - V_j - T))$$

as a local section generator.

The finite group G acts on the variety $\mathfrak{X}_0$. We can modify the gluing along D_{ij} by $\lambda \in \operatorname{Aut}^0(D^0_{ij})$ which is compatible with the action G such that $\mathcal{O}_E(-\mathfrak{X}_0)$ has the gluing data at a triple point t_{ijk},

$$\begin{cases} z_i z_j z_k \in H^0(\mathcal{O}_{D_{ik}}(-V_i - V_k - T)); \\ z_i z_j z_k \in H^0(\mathcal{O}_{D_{jk}}(-V_j - V_k - T)); \\ \lambda^{-1} z_i z_j z_k \in H^0(\mathcal{O}_{D_{ij}}(-V_i - V_j - T)). \end{cases}$$

At the triple point t_{ijl}, the formula is similar. Now look at the exact sequence (3.5.2), and we have

$$\left(\mathcal{O}^*_{\widetilde{E}}/\mathcal{O}^*_E\right)_{t_{ijk}} \cong (\mathbb{C}^*)^3/\mathbb{C}^*.$$

If $(\mathbb{C}^*)^3$ has basis (e_{ij}, e_{jk}, e_{ik}) and the action $\mathbb{C}^*$ is the diagonal subspace, then by the gluing, the effect on $\mathcal{O}_E(\mathfrak{X}_0)$ is to multiply the e_{ij} component at t_{ijk} by λ and the corresponding component at t_{ijl} by λ^{-1}. This is exactly the action of G_{ij} on $\mathrm{Pic}^0(E)$, up to a power of 2. Thus, we have $\mathcal{O}_E(\mathfrak{X}_0) = \mathcal{O}_E$. □

For the d-semi-stable G-variety $\mathfrak{X}_0$, let

$$\pi : \widetilde{\mathfrak{X}}_0 \to \mathfrak{X}_0$$

be the normalization. Let $\widetilde{T} \to T$ and $\widetilde{Q} \to Q$ be the corresponding normalizations of the locus T and Q. Since $\mathfrak{X}_0$ is a variety with normal crossings, [5, (3.2), (3.3)] implies that there exists an intrinsically defined subsheaf

$$\Lambda^1_{\mathfrak{X}_0} \subset \pi_*\Omega^1_{\widetilde{\mathfrak{X}}_0}(\log \widetilde{Q})$$

and a resolution

$$0 \to \Omega^1_{\mathfrak{X}_0}/\tau_{\mathfrak{X}_0} \to \Lambda^1_{\mathfrak{X}_0} \to \pi_*\mathcal{O}_{\widetilde{Q}} \to \pi_*\mathcal{O}_{\widetilde{T}} \to 0$$

where $\widetilde{T} = T$, $\tau_{\mathfrak{X}_0}$ is the torsion point of $\Omega^1_{\mathfrak{X}_0}$. Here the sheaf $\Lambda^1_{\mathfrak{X}_0}$ is intrinsic such that $\Lambda^2\Lambda^1_{\mathfrak{X}_0} \cong \omega_{\mathfrak{X}_0}$. Choose a generating section $\xi \in H^0(T^1_{\mathfrak{X}_0})$, and via Lie bracket, we have the map

$$[\cdot, \xi] : T^0_{\mathfrak{X}_0} \to T^1_{\mathfrak{X}_0}. \tag{3.5.3}$$

We have that

$$S_{\mathfrak{X}_0} := \ker([\cdot, \xi]) \cong (\Lambda^1_{\mathfrak{X}_0})^*.$$

The same proof in [7, Lemma 2.7] gives $H^0(\mathfrak{X}_0, \Lambda^1_{\mathfrak{X}_0}) = 0$. We have the following results as in [7, Lemma 2.8]:

(1) $H^2(T^0_{\mathfrak{X}_0}) = 0$;
(2) The natural map $\mathbb{T}^1_{\mathfrak{X}_0} \to H^0(T^1_{\mathfrak{X}_0})$ is surjective;
(3) The natural map $H^1(T^0_{\mathfrak{X}_0}) \otimes H^0(T^1_{\mathfrak{X}_0}) \to H^1(T^1_{\mathfrak{X}_0})$ is surjective.

The first one is from the following resolution:

$$0 \to \Omega^1_{\mathfrak{X}_0}/\tau_{\mathfrak{X}_0} \to \pi_*\Omega^1_{\widetilde{\mathfrak{X}}_0} \to \pi_*\Omega^1_{\widetilde{Q}} \to 0.$$

We have $H^0(V_0, \Omega^1_{V_0}) = 0$, see [7, (1.5.3)], which implies that $H^0(\Omega^1_{\mathfrak{X}_0}/\tau_{\mathfrak{X}_0}) = 0$. Serre duality implies that

$$H^2(T^0_{\mathfrak{X}_0}) \cong H^0(\Omega^1_{\mathfrak{X}_0}/\tau_{\mathfrak{X}_0} \otimes \omega_{\mathfrak{X}_0})^*.$$

By the construction for $\mathfrak{X}_0$, we have

$$\pi^*\omega_{\mathfrak{X}_0}|_{V_0} = \mathcal{O}_{V_0}(-D)$$

and

$$\pi^*\omega_{\mathfrak{X}_0}|_{V_i} = \mathcal{O}_{V_i}, \quad i > 0,$$

therefore, $H^0(F \otimes \omega_{\mathfrak{X}_0}) \subset H^0(F)$ for any torsion free coherent sheaf F. Thus, we get $H^2(T^0_{\mathfrak{X}_0}) = 0$ since $H^0(\Omega^1_{\mathfrak{X}_0}/\tau_{\mathfrak{X}_0}) = 0$.

(2) comes from the Ext spectral sequence

$$\mathbb{T}^i_{\mathfrak{X}_0} = \bigoplus_{p+q=i} H^p(\mathfrak{X}_0, \mathcal{E}xt^q(\Omega^1_{\mathfrak{X}_0}, \mathcal{O}_{\mathfrak{X}_0})).$$

(3) is from (3.5.3), since we have

$$0 \to S_{\mathfrak{X}_0} \to T^0_{\mathfrak{X}_0} \stackrel{[\cdot,\xi]}{\to} T^1_{\mathfrak{X}_0} \to 0.$$

It is enough to show that $H^2(S_{\mathfrak{X}_0}) = 0$, or equivalently $H^0(\Lambda^1_{\mathfrak{X}_0} \otimes \omega_{\mathfrak{X}_0}) = 0$ which is true since $H^0(\Lambda^1_{\mathfrak{X}_0}) = 0$.

Now it is ready to prove the theorem. The proof is the same as in [5, (5.10)]. We assume that the sheaf $(T^1_{\mathfrak{X}_0})^G \neq 0$, which implies that the G-equivariant deformations exist.

The result then follows from a basic argument in the deformation and obstruction theory. First let

$$\mathbb{T}^1_{\mathfrak{X}_0} \otimes \mathbb{T}^1_{\mathfrak{X}_0} \to \mathbb{T}^2_{\mathfrak{X}_0}$$

be the Lie bracket map. Since $H^2(T^0_{\mathfrak{X}_0}) = 0$, the Lie bracket $\mathbb{T}^1_{\mathfrak{X}_0} \otimes \mathbb{T}^1_{\mathfrak{X}_0} \to \mathbb{T}^2_{\mathfrak{X}_0}$ induces

$$[\cdot,\cdot] : H^1(T^0_{\mathfrak{X}_0}) \otimes H^0(T^1_{\mathfrak{X}_0}) \to H^1(T^1_{\mathfrak{X}_0}).$$

As in [7, (5.10)], we let

$$W_1 = H^0(T^1_{\mathfrak{X}_0}) \subset \mathbb{T}^1_{\mathfrak{X}_0},$$

a hyperplane since $\mathbb{T}^1_{\mathfrak{X}_0} = H^0(T^1_{\mathfrak{X}_0}) \oplus H^1(T^0_{\mathfrak{X}_0})$ and $H^0(T^1_{\mathfrak{X}_0}) \cong \mathbb{C}$. Let $e \in \mathbb{T}^1_{\mathfrak{X}_0}$ be mapped to $1 \in H^0(T^1_{\mathfrak{X}_0}) \cong \mathbb{C}$, and

$$W_2 = \{v \in \mathbb{T}^1_{\mathfrak{X}_0} | [v,e] = 0\} = \{x + \lambda e | \lambda \in \mathbb{C}, x \in W_1, [x,e] = 0\}.$$

Then $W_1 \cap W_2 = \{x \in W_1 | [x,e] = 0\}$ is a hyperplane in W_2. By the basic deformation theory, we have a holomorphic map:

$$f : \mathbb{T}^1_{\mathfrak{X}_0} \to \mathbb{T}^2_{\mathfrak{X}_0}$$

such that $f(0) = 0$, f has no linear terms, and $f^{-1}(0)$ is the base space of a versal deformation of $\mathfrak{X}_0$. As in [5, (5.10)], $f^{-1}(0)$ contains the smooth divisor $N_1 \subset W_1$ which corresponds to local trivial deformations. Then from [5, (5.10)],

$$f^{-1}(0) = N_1 \cup N_2$$

where $N_2 = \{h(v) = 0\}$ for

$$h : (\mathbb{T}^1_{\mathfrak{X}_0}, 0) \to (\mathbb{T}^2_{\mathfrak{X}_0}, 0)$$

such that $f = g \cdot h$, and $\{g = 0\}$ is the reduced germ of N_1. Then N_1 corresponds to the local trivial deformations of $\mathfrak{X}_0$, and $N_2 - N_1$ corresponds to smooth rational surface (Y, D), which is from [5, (2.5)]. Thus, the deformation theory implies that we have that the G-surface $\mathfrak{X}_0$ admits a smoothing $\pi : \mathfrak{X} \to \Delta$ satisfying the conditions in the theorem.

□

4. Construction of Type III canonical degeneration pair

4.1. **Integral-affine surface.** We recall the integral-affine surfaces in [8], [4, §3].

A basis triangle of $\mathbb{R}^2$ is a triangle Δ of area $\frac{1}{2}$ with integral vertices in $\mathbb{Z}^2 \subset \mathbb{R}^2$. Any two pairwise edges of a basis triangle form a basis for $\mathbb{Z}^2$.

Definition 4.1. *(*[4, Definition 3.1]*) A triangulated integral-affine surface with singularities is a triangulated real surface S, possibly with boundary such that*

(1) the complement of the vertices $\{v_i\} \subset S$ of the triangulation admits an atlas of charts into $\mathbb{R}^2$, whose transition functions take values in $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$.
(2) the interior of every triangle admits a chart to a basis triangle.

An integral-affine surface with singularities has a canonical orientation induced from the standard orientation on $\mathbb{R}^2$. Let e_{ij} be the edge $v_i - v_j$ in the triangulation of S. Let f_{ijk} be the triangle whose counterclockwise ordered vertices are v_i, v_j, v_k. In this chart we can write $e_{ij} = v_j - v_i$.

Let S be a triangulated real surface by basis triangles. The boundary $\partial S = P_1 + \cdots + P_n$ is a polygon, where each P_i is integral-affine and is a line segment between two lattice points. We assume that ∂S is maximal which means the union of two distinct boundary components is never integral-affine equivalent to a single line segment.

Definition 4.2. *If the atlas of integral-affine charts on $S - \{v_i\}$ extends to all vertices $\{v_i\}$, then we say S is non-singular. Otherwise S is singular. Let S_{sing} denote the singular vertices, i.e., the vertices which the integral-affine structure fails to extend.*

Remark 4.3. *Let f_{ijk} be a triangle formed by v_i, v_j, v_k in the counterclockwise direction. Let $v_i - v_l$ be another edge such that v_i, v_k, v_l form another triangle f_{ikl} in the counterclockwise direction again. We define the self-intersection number d_{ik} by*

$$d_{ik} e_{ik} = e_{ij} + e_{il}.$$

From [4, Proposition 3.6, Proposition 3.7], $d_{ik} + d_{ki} = 2$ *for every interior edges e_{ik}. Also a triangulated integral-affine surface S is uniquely determined by the data of a collection of negative self-intersections d_{ik} for each directed interior edge e_{ik} such that $d_{ik} + d_{ki} = 2$.*

Definition 4.4. *Let (Y, D) be a Looijenga pair. The pseudo-fan of (Y, D) is a triangulated integral-affine surface whose underlying surface $S_{(Y,D)}$ is the cone over the dual complex of D.*

Let e_i be the edge from the cone point to the vertex corresponding to D_i. Then the negative self-intersection of e_i is:

$$d_i = \begin{cases} -D_i^2, & n > 1; \\ 2 - D_i^2, & n = 1. \end{cases}$$

Also from [4, Proposition 3.9], the integral-affine structure on the pseudo-fan of (Y, D) extends to the cone point if and only if (Y, D) is a toric pair. Here we recall that a toric Looijenga pair $(Y^{\text{toric}}, D^{\text{toric}})$ is a toric surface Y^{toric} such that D^{toric} is its toric boundary.

For a Type III canonical degeneration pair $\mathfrak{X}_0$, the dual complex $\Gamma(\mathfrak{X}_0)$ is a triangulation of the sphere S^2. The vertices $\{v_i\}$ correspond to the components

V_i, the directed edges e_{ij} correspond to double curves D_{ij}, and triangular faces f_{ijk} correspond to triple points in Remark 4.3.

From [4, Proposition 3.10], the dual complex $\Gamma(\mathfrak{X}_0)$ has a triangulated integral-affine structure such that

$$d_{ij} := \begin{cases} -D_{ij}^2, & \ell(D_i) \geq 2; \\ 2 - D_{ij}^2, & \ell(D_i) = 1 \end{cases}$$

where d_{ij} is the negative self-intersection of e_{ij}. Moreover, the integral-affine structure extends maximally to $\Gamma(\mathfrak{X}_0) - (\{v_i | Q(V_i, D_i) > 0\} \cup \{v_0\})$.

4.2. **Surgeries.** Let us recall the surgeries on the integral-affine surface in [4, §4]. The surgeries on the integral-affine surface are motivated by the almost toric fibration in [25]. It is a generalization of the moment map from toric surfaces to its moment polygon S.

Let S be a singular integral-affine surface which is homeomorphic to a disc, and we let

$$\partial S = P_1 + \cdots + P_n$$

is the union of a sequence of segments P_i such that each segment integral-affine equivalent to a straight line segment between two lattice points. The boundary components P_i go counterclockwise around S when i increases. Denote by

$$v_{i,i+1} = P_i \cap P_{i+1}$$

the vertex, and let x_i, y_i be the primitive integral vectors emanating from $v_{i,i+1}$ along P_{i+1} and P_i, respectively. Then we have $y_{i+1} = -x_i$. As in [4, Definition 4.2], we define negative self-intersection d_i of P_i by:

$$d_i y_i = y_{i-1} - x_i = y_{i-1} + y_{i+1}.$$

If $\mu : (Y, D, \omega) \to S$ is an almost toric fibration, then it is a Lagrangian fibration whose general fiber is a smooth 2-torus, which degenerates under symplectic reduction, over the boundary ∂S. Also the interior fibers may also degenerate to necklaces of spheres at some finite set of points.

There are two type of surgeries on S.

4.2.1. *Internal blow-up.* The first one is the internal blow-up of S on the boundary P_i. The surgery is given by:

Step I: Delete the triangle $T \subset S$ which satisfies the properties:

(1) One edge e_T of T is proper subsegment of P_i;
(2) $T \setminus e_T \subset S - S_{\text{sing}}$ belongs to the interior part of $S - S_{\text{sing}}$;
(3) T is an integer multiple n size of a basis triangle.

Step II: Let v be the unique vertex of T lying in the interior of S, and let (e_1, e_2) be the oriented lattice basis emanating from v along the edges of T. The glue the edge e_2 of $S - T$ to the edge along e_1 of $S - T$ via the unique affine-linear map which fixes v, and maps $e_2 \mapsto e_1$, and preserving the line containing P_i.

The resulting integral-affine surface is an internal blow-up of S on P_i. The singular set is $S_{\text{sing}} \cup \{v\}$ and n is the size of the surgery. Please see [4, Figure 3].

4.2.2. *Node-smoothing.* The second one is the node-smoothing of S at the node $P_i \cap P_{i+1}$. The surgery is:

At the node $P_i \cap P_{i+1}$, for $n \in \mathbb{N}$, cut a segment from $v_{i,i+1}$ to

$$v := v_{i,i+1} + n(x_i + y_i)$$

lying in $S - \partial S$. Then we glue the clockwise edge of the cut to the counterclockwise edge of the cut by the shearing map which point-wise fixes the line containing the cut and maps x_i to the $-y_i$.

The resulting integral-affine surface is the node smoothing at $P_i \cap P_{i+1}$, and has size n. The singular set is $S_{\text{sing}} \cup \{v\}$.

4.2.3. *Surgeries and self-intersection numbers.* Similar to Proposition 3.6, an internal blow-up of the integral-affine surface S on the boundary P_i changes the negative self-intersections of the boundary components by:

$$(\cdots, d_i, \cdots) \mapsto (\cdots, d_i + 1, \cdots).$$

A node smoothing at $P_i \cap P_{i+1}$ of S changes the negative self-intersections of the boundary components by:

$$(\cdots, d_i, d_{i+1}, \cdots) \mapsto (\cdots, d_i + d_{i+1} - 2, \cdots).$$

Now suppose that we have an integral-affine disc such that the adjacent edges of ∂S meet to form lattice bases, and the negative self-intersections of $P_i \subset \partial S$ are:

$$\begin{cases} d_i \geq 2, & \text{for all } i; \\ d_i \geq 3, & \text{for some } i. \end{cases}$$

Then from [4, Proposition 4.6], there is a natural embedding $S \hookrightarrow \hat{S}$ where $\hat{S}$ is an integral-affine sphere and $\hat{S}_{\text{sing}} = S_{\text{sing}} \cup \{v_0\}$ for a distinguished point $v_0 \in \hat{S} \setminus S$.

From [4, Remark 4.8, Definition 4.7], $v_0 \in \hat{S}$ may not be integral. Since it is rational, we take the order k refinement $S[k]$, and $\hat{S}[k] = S[k] \cup C[k]$, where $C := \hat{S} \setminus S$. Thus $v_0 \in \hat{S}[k]$.

4.3. **The construction.** Now we are ready to construct a Type III canonical degeneration pair from a Looijenga pair (Y, D), together with a finite group G-action. We first have:

Proposition 4.5. *The Looijenga pair (Y, D) can be represented by a sequence of G-equivariant node smoothings and G-equivariant internal blow-ups from a G-minimal pair.*

Proof. For the Looijenga pair (Y, D), the finite group G, taken as automorphism subgroup, lies in $\mathrm{Aut}_{\mathbb{C}}(D \subset Y)$, thus, from [1, Theorem 0.1], the internal blow-up and corner blow-up in §4.2.1 and can be made into G-equivariant by the G-equivariant modifications. For the node-smoothing in §4.2.2, geometrically it is like the smoothing of the singularity $(xy = 0)$, and this process can be made into G-equivariant by a suitable action of G on the node.

Thus, as a pair, we know (Y, D) can be given by a sequence of internal blow-ups and corner blow-ups from a minimal pair, i.e., we have

$$(Y, D) \xrightarrow{\alpha} (Y_1, D_1) \xrightarrow{\beta} (Y_0, D_0)$$

where α consists of internal blow-ups and β consists of corner blow-ups. $\square$

Now suppose that we have a Looijenga pair (Y, D), together with a finite group G-action such that G acts freely on the complement $Y - D$. We also require that the cycle D is negative-definite.

We perform the arguments in [4, §5] to construct a Type III canonical degeneration pair $\mathfrak{X}_0 = \cup_i V_i$, such that there exists a G-action on V_0. First for a Looijenga pair (Y, D) with a G-action, there exist a sequence of G-equivariant internal blow-ups and corner blow-ups to a minimal pair

$$(Y, D) \to (Y^{\min}, D^{\min}).$$

We can forget about the G-action so that the minimal pair has a toric model

$$(Y^{\min}, D^{\min}) \to (Y^{\text{toric}}, D^{\text{toric}}).$$

Let S^{toric} be the moment polygon of the toric model $(Y^{\text{toric}}, D^{\text{toric}})$, then we perform the internal blow-ups and node smoothing as in §4.2 for S^{toric} to get the integral-affine surface S for the Looijenga pair (Y, D). Here the argument is the same as in [4, §4] since we don't need the G-action on the rational surfaces V_i for $i > 0$. From the argument, there are totally $Q(Y, D)$ surgeries of fixed sizes.

We can complete the integral-affine surface S to a sphere $\hat{S}$ as in [4, Proposition 4.6]. We also take an order k-refinement $\hat{S}[k]$ such that $v_0 \in \hat{S} \setminus S$ is integral. The refinement $\hat{S}[k]$ admits a triangulation into basis triangles.

Note that there may exist many such triangulations and we choose the one that attains the minimal number of edges emanating from v_0.

For each $v_i \in \hat{S}[k]$, if v_i is non-singular, then $\text{Star}(v_i)$ is the pseudo-fan of a toric surface pair (V_i, D_i). This toric surface may not admit a G-action, but we do not need this. Suppose that we have a vertex $v_i \in \hat{S}[k]_{\text{sing}}$ which is singular, $v_i \neq v_0$. Recall such a vertex v_i is given by a surgery on S^{toric}. Let $v_i^{\text{toric}} \in S^{\text{toric}}$ be the preimage of this vertex under the surgery. Then we have

An internal blow-up on S^{toric} corresponds to a node smoothing on $\text{Star}(v_i^{\text{toric}})$.

A node smoothing on S^{toric} corresponds to an internal blow-up on $\text{Star}(v_i^{\text{toric}})$. Please see [4, Figure 8] for the graph.

Thus there exists a Looijenga pair (V_i, D_i) with pseudo-fan $\text{Star}(v_i)$. For $v_0 \in \hat{S}[k]$, [4, Lemma 5.3] showed that $\text{Star}(v_0)$ is the pseudo-fan of (V_0, D'). From the automorphism group explanation as in §2.2, the finite group G also acts on the Inoue-Hirzebruch surface V_0, so that these two dual cusps D, D' are all contractible. This is exactly what we want for the G-action on V_0.

So let

$$\mathfrak{X}_0 := \bigcup_{v_i \in \hat{S}[k]} (V_i, D_i)$$

where we identify D_{ij} with D_{ji} to make the nodes of D_i are identified with the nodes of D_j. It is routine to check that the triple formula holds so that $\mathfrak{X}_0$ is a Type III anti-canonical pair and there exists a G-action on the Inoue-Hirzebruch surface V_0. There are several modifications for the construction of P. Engel in [4, §5.4, §5.5, §5.6] that we do not need to discuss.

5. The main result and examples

5.1. **The proof of Theorem 1.2.** We prove Theorem 1.2. Suppose the surface cusp singularity $(\overline{W}, q')$ admits a smoothing such that it is induced from the G-equivariant smoothing of the cusp $(\overline{V}, p')$. We let

$$\pi : \overline{\mathcal{V}} \to \Delta$$

be the G-equivariant smoothing of the cusp $(\overline{V}, p')$. Recall that in §2.2, the G-Inoue-Hirzebruch surface $(\overline{\overline{V}}, p, p')$ and (V, D, D'). We may construct the smoothing $\pi : \overline{\mathcal{V}} \to \Delta$ as follows: first we take the G-equivariant smoothing of $(\overline{\overline{V}}, p')$,

$$\pi : \overline{\overline{\mathcal{V}}} \to \Delta$$

such that each fiber $\overline{\overline{\mathcal{V}}}_t$ contains a cusp singularity p_t for $t \neq 0$. Therefore, we take the G-equivariant simultaneous resolution of singularities of $\{p_t\}$ to obtain

$$\pi : \overline{\mathcal{V}} \to \Delta$$

such that the generic fiber $\overline{\mathcal{V}}_t = (Y_t, D_t)$ is a rational surface Y_t, together with an anti-canonical divisor D_t. This is a G-Looijenga pair. Since G acts on the family $\pi : \overline{\mathcal{V}} \to \Delta$, there exists a subgroup $H \subseteq G$ which acts effectively on the fiber $\overline{\mathcal{V}}_t = (Y_t, D_t)$. From the construction in §2.2 again, the group H acts on $Y_t \setminus D_t$ freely and D_t is negative-definite which can be contracted to the cusp singularity p_t. The quotient $(Y_t \setminus D_t)/H$ is still an Inoue-Hirzebruch surface. We can take resolution of singularities for Y_t/H again such that the quotient is a smooth Looijenga pair (X, E).

Conversely, suppose that there is a negative definite Looijenga pair (Y, D), together with a finite group G-action such that after possible resolution of singularities the quotient $(Y, G)/G$ becomes a Looijenga pair (X, E), we need to show that the dual cusp p' of the cusp D admits a G-equivariant smoothing which induces a smoothing of the dual quotient cusp q' of q corresponding to E.

From the construction in §4.3, we have the following Type III canonical degeneration pairs from a G-Looijenga pair (Y, D):

$$\mathfrak{X}_0 = \bigcup_{i \in I, i \geq 0} (V_i, D_i)$$

such that (V_0, D, D') is an Inoue-Hirzebruch surface. Also there is a G-action on the surface V_0 such that if $(V, D, D') \to (\overline{\overline{V}}_0, p, p')$ is the contraction of D, D' to p, p', then the G-action on $\overline{\overline{V}}_0$ only has two fixed points p, p'.

Let $\mathfrak{X} \to \Delta$ be the deformation of $\mathfrak{X}_0$. From [24], all the components $\sum_{i \geq 1}(V_i, D_i)$ is contractible, and we get

$$\begin{tikzcd} \mathfrak{X} \arrow[rr] \arrow[dr, "\pi"'] & & \overline{\mathfrak{X}} \arrow[dl, "\overline{\pi}"] \\ & \Delta & \end{tikzcd}$$

such that $\overline{\mathfrak{X}} \to \Delta$ is a deformation of $(\overline{V}_0, D, p')$ which is a G-equivariant smoothing of the cusp p'.

Note that the fiber (Y_t, D_t) of the $\overline{\pi} : \overline{\mathfrak{X}} \to \Delta$ for $t \neq 0$ admits a G-action such that the G-action is free on $Y_t - D_t$, and (Y_t, D_t) is negative-definite. Then we simultaneously contract such D_t's and get

$$\overline{\overline{\pi}} : \overline{\overline{\mathfrak{X}}} \to \Delta$$

which is a smoothing of $(\overline{\overline{V}}_0, p, p')$.

Then we take the quotient $\pi : \overline{\overline{\mathcal{X}}} = \overline{\overline{\mathfrak{X}}}/G \to \Delta$ such that it is a smoothing of $\overline{\overline{W}}_0 = \overline{\overline{V}}_0/G$, and this is another Inoue-Hirzebruch surface $(\overline{\overline{W}}_0, q, q')$. We take simultaneously resolution of singularities for $q_t \in \overline{\overline{\mathcal{X}}}_t$ and get

$$\overline{\mathcal{X}} \to \Delta$$

which is a smoothing of $(\overline{W}_0, E, q')$ and the fiber (X_t, E_t) is a Looijenga pair. From Proposition 3.11, the pair (X_t, E_t) is the quotient of the pair (Y_t, D_t) for $t \neq 0$. This gives the smoothing of the cusp q'.

5.2. **Example 1.** We consider an example. Let $(\widetilde{V}, p')$ be a negative definite cusp singularity whose resolution cycle (in terms of negative self-intersection numbers) is $\mathbf{d}' = (5, 2)$. This is a hypersurface cusp given by

$$\{x^3 + y^3 + z^5 + xyz = 0\}.$$

From [20, Page 308, Example], this cusp admits a $G = \mathbb{Z}_2$-action whose quotient is a cusp (V, p) which is also a hypersurface cusp whose resolution cycle is given by (8).

The dual cusp p of (V, p') is given by $\mathbf{d} = (3, 2, 2, 2, 2, 2)$; and the dual cusp of $(\widetilde{V}, p')$ is given by $(4, 2, 2)$. Thus, we have the following diagram:

$$\begin{tikzcd}
(\widetilde{V}, p') \arrow[r, leftrightarrow] \arrow[d] & (Y, D) \arrow[d] \\
(V, p') \arrow[r, leftrightarrow] & (X, E)
\end{tikzcd}$$

where (Y, D) is the hyperbolic Looijenga pair with negative self-intersection sequence $(4, 2, 2)$, and (X, E) is the Looijenga pair with E and the negative self-intersection sequence is given by $\mathbf{d} = (3, 2, 2, 2, 2, 2)$.

From Proposition 3.6, we perform corner blow-ups and do the following:

$$(4, 2, 2) \to (3, 1, 1, 2) \to (2, 2, 0, 1, 2) \to (2, 1, 0, 1, 1, 1).$$

The cycle $(2, 1, 0, 1, 1, 1)$ corresponds to a toric pair, since if D^{toric} is the divisor corresponding to the negative self-intersection sequence $(2, 1, 0, 1, 1, 1)$, then $Q(D^{\text{toric}}) = 0$.

We do the same process by internal blow-down:

$$(3, 2, 2, 2, 2, 2) \to (2, 1, 1, 1, 1, 0).$$

Then in the toric model $(X^{\text{toric}}, E^{\text{toric}})$, $E^{\text{toric}})$ is given by $(2, 1, 1, 1, 1, 0)$. We can see that the $\mathbb{Z}_2$ acts on $(Y^{\text{toric}}, D^{\text{toric}})$ by permutation on the components of D^{toric} to get E^{toric}.

5.3. **Example 2.** Here is another example. Let (V, p') be a negative definite cusp singularity whose resolution cycle (in terms of negative self-intersection numbers) is $\mathbf{d}' = (6, 9)$, which is not a complete intersection cusp since $(6-2)+(9-2) = 11 > 4$. The charge $Q(D') = 12 + (6-3) + (9-3) = 21$. Then from Proposition 3.5, this cusp D' has a rational dual D with negative self-intersection sequence $\mathbf{d} = (3, 2, 2, 2, 3, 2, 2, 2, 2, 2, 2)$.

From [18, Proof of Theorem 4.1], $t = 53 - 1 = 52$, and the finite cover $(\widetilde{V}, p') \to (V, p')$ is a hypersurface cusp whose resolution cycle is given by $(3, \underbrace{2, 2, \cdots, 2}_{49})$.

From [17, Lemma 2.5], this cusp is given by

$$\{x^2 + y^3 + z^{56} + xyz = 0\}.$$

Let G be the transformation group of the cover $(\widetilde{V}, p') \to (V, p')$. First we have the exact sequence

$$0 \to H \rtimes \mathbb{Z} \to \pi_1(\Sigma) \to G' \to 0$$

where $H \subset \pi_1(\Sigma) = \mathbb{Z}^2 \rtimes \mathbb{Z}$ is generated by $(0, 1), (-1, 9)$. Then

$$G = G' \rtimes H_1(\Sigma, \mathbb{Z})_{\mathrm{tor}}.$$

The dual cusp p of (V, p') is given by $\mathbf{d} = (3, 2, 2, 2, 3, 2, 2, 2, 2, 2, 2)$; and the dual cusp of $(\widetilde{V}, p')$ is given by (52). Thus, we have the following diagram:

$$\begin{tikzcd}
(\widetilde{V}, p') \arrow[r, leftrightarrow] \arrow[d] & (Y, D) \arrow[d] \\
(V, p') \arrow[r, leftrightarrow] & (X, E)
\end{tikzcd}$$

where (Y, D) is the hyperbolic Looijenga pair with $D^2 = -52$, and (X, E) is the Looijenga pair with E given by $(3, 2, 2, 2, 3, 2, 2, 2, 2, 2, 2)$.

From [4, Figure 12], the Type III anti-canonical pairs of $(6, 9)$ and $(3, 2, 2, 2, 3, 2, 2, 2, 2, 2, 2)$ are given. Similar way we can get the Type III anti-canonical pairs of $(3, \underbrace{2, 2, \cdots, 2}_{49})$ and (52), whose quotient under G gives the above Type III anti-canonical pairs, up to resolution of singularities.

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