THE VAFA-WITTEN INVARIANTS VIA SURFACE DELIGNE-MUMFORD STACKS II: ROOT STACKS AND PARABOLIC HIGGS PAIRS

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ABSTRACT. Let S be a smooth projective surface, and $D \subset S$ a smooth divisor. In this paper we prove that the moduli space of stable Higgs sheaves on a d-th root stack $\mathcal{S}$ associated with a pair (S, D) is isomorphic to the moduli space of parabolic Higgs sheaves on (S, D). We also construct and generalize the moduli space of relative stable sheaves on (S, D) of Kapranov, the moduli space of relative with the stacky divisor of stable Higgs sheaves on $\mathcal{S}$, and prove that it is related to the geometric Eisenstein series associated with the curve D. We relate the Tanaka-Thomas's Vafa-Witten invariants for the root stack $\mathcal{S}$ to the parabolic situation considered by Kapranov inspired by the S-duality conjecture.

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1. Introduction

In this paper we continue to study the Tanaka-Thomas's Vafa-Witten invariants for projective surface Deligne-Mumford stacks [48], [49], [25] in the case of root stacks over a smooth projective surface. The definition of the twisted Vafa-Witten invariants for cyclic gerbes $\mathcal{S} \to S$ in [27] were used to define the Langlands dual gauge group $\mathrm{SU}(r)/\mathbb{Z}_r$-Vafa-Witten invariants and prove the S-duality conjecture of Vafa-Witten [54] for $\mathbb{P}^2$ and K3 surfaces. For the root stack $\mathbb{P}(1,2,2)$, in [26], the Vafa-Witten invariants were calculated using the result in [13], and it also can give the formula inspired by the S-duality conjecture. So it is generally interesting to see how the root stack is related to the S-duality conjecture.

1.1. **Background.**

The S-duality conjecture of Vafa-Witten [54] predicted that the generating function of the Euler characteristic of the moduli space of stable coherent sheaves on projective surfaces should be modular forms. They also conjecture a S-transformation formula for the invariants of counting instantons for the gauge group $\mathrm{SU}(r)$ and its Langlands dual $\mathrm{SU}(r)/\mathbb{Z}_r$. In physics the S-duality is a very rich conjecture, for instance in [36] Kapustin-Witten related the S-duality to Langland duality in number theory. In the mathematics side, the moduli space of solutions of the Vafa-Witten equation on a projective surface S has a partial compactification by Gieseker semistable Higgs pairs (E, ϕ) on S, where E is a coherent sheaf with rank > 0, and $\phi \in \mathrm{Hom}_S(E, E \otimes K_S)$ is a section called a Higgs field.

The formulas in [54] (for instance Formula (5.38) of [54]) and some mathematical calculations as in [14], implies that the invariants in [54] may have other contributions except purely from the surfaces. In [48], [49] Tanaka and Thomas define the Vafa-Witten invariants using the moduli space $\mathcal{N}$ of Gieseker semi-stable Higgs pairs (E, ϕ) on S with topological data (r, c_1, c_2), where $r =$ rank of the torsion free sheaf E, and c_1, c_2 are the first and second Chern classes of E. In [25], Kundu and the first author generalized the definition and construction of Tanaka-Thomas to smooth projective two dimensional Deligne-Mumford stacks, which we call surface DM stacks.

Let $\mathcal{S}$ be a surface DM stack and $\mathcal{X} := \mathrm{Tot}(K_{\mathcal{S}})$ the total space of the canonical line bundle of $\mathcal{S}$. Again by spectral theory, the abelian category of Higgs pairs on S is equivalent to the abelian category of two dimensional torsion sheaves on $\mathcal{X}$ supported on $\mathcal{S}$. By choosing suitable generating sheaf Ξ for S and a modified Hilbert polynomial H as in [46], the moduli space $\mathcal{N}^H(\mathcal{S})$ of Gieseker semi-stable Higgs pairs (E, ϕ) on S with fixed modified Hilbert polynomial H is isomorphic to the moduli space of Gieseker semi-stable torsion sheaves $\mathcal{E}_\phi$ on the total space $\mathcal{X}$ with the fixed corresponding modified Hilbert polynomial. The modified Hilbert polynomial is determined by a K-group class $\mathbf{c} \in K_0(\mathcal{S})$. We denote by $\mathcal{N} := \mathcal{N}^{\mathbf{c}}(\mathcal{S})$. Since $\mathcal{X}$ is a smooth Calabi-Yau threefold DM stack, the moduli space $\mathcal{N}$ admits a symmetric obstruction theory in [2]. Therefore there exists a dimension zero virtual fundamental cycle $[\mathcal{N}]^{\mathrm{vir}} \in H_0(\mathcal{N})$. The moduli space $\mathcal{N}$ is not compact, but it admits a $\mathbb{C}^*$-action induced by the $\mathbb{C}^*$-action on $\mathcal{X}$ by scaling the fibres of $\mathcal{X} \to \mathcal{S}$. The $\mathbb{C}^*$-fixed locus $\mathcal{N}^{\mathbb{C}^*}$ is compact, then from [12], $\mathcal{N}^{\mathbb{C}^*}$ inherits a perfect obstruction theory from $\mathcal{N}$ and the virtual localized invariant

$$\widetilde{\mathrm{VW}}_{\mathbf{c}}(\mathcal{S})=\int_{[\mathcal{N}^{\mathbb{C}^*}]^{\mathrm{vir}}}\frac{1}{e(N^{\mathrm{vir}})} \tag{1.1.1}$$

is defined as the $U(r)$ Vafa-Witten invariants, where N^{vir} is the virtual normal bundle.

But in this case the obstruction sheaf has a trivial summand making the invariants vanishes for most of the surfaces. Then in [48] and the stacky version [25], one considers the moduli space $\mathcal{N}_L^{\perp}$ of stable Higgs sheaves (E,ϕ) with fixed determinant L and trace free ϕ. Then the space $\mathcal{N}_L^{\perp}$ also admits a symmetric obstruction theory and there exists a $\mathbb{C}^*$-action induced by the $\mathbb{C}^*$-action on $\mathcal{X}$ by scaling the fibres. The $\mathbb{C}^*$-fixed locus $(\mathcal{N}_L^{\perp})^{\mathbb{C}^*}$ is compact, then the virtual localized invariant

$$\mathrm{VW}_{\mathbf{c}}(\mathcal{S})=\int_{[(\mathcal{N}_L^{\perp})^{\mathbb{C}^*}]^{\mathrm{vir}}}\frac{1}{e(N^{\mathrm{vir}})} \tag{1.1.2}$$

is defined as the $SU(r)$ Vafa-Witten invariants since the gauge group of the associated sheaves is $SU(r)$, where N^{vir} is also the virtual normal bundle.

There is another invariants defined by the weighted Euler characteristic

$$\mathrm{vw}(\mathcal{S})=\chi(\mathcal{N}_L^{\perp},\nu_N) \tag{1.1.3}$$

where $\nu_N:\mathcal{N}_L^{\perp}\to\mathbb{Z}$ is the Behrend function, see [2]. We call it the small Vafa-Witten invariant. If $\mathcal{N}_L^{\perp}$ is compact and admits a symmetric obstruction theory, then

$$\int_{[\mathcal{N}_L^{\perp}]^{\mathrm{vir}}}1=\chi(\mathcal{N}_L^{\perp},\nu_N)$$

and this is Behrend's theorem in [2, Theorem 4.18]. On the other hand the $\mathbb{C}^*$ action on $\mathcal{N}_L^{\perp}$ induces a cosection $\mathcal{O}_{\mathcal{N}_L^{\perp}}\to\mathcal{O}_{\mathcal{N}_L^{\perp}}$ by differentiating the $\mathbb{C}^*$-action. Then there is a Kiem-Li localized cycle

$$[\mathcal{N}_L^{\perp}]^{\mathrm{vir}}_{\mathrm{loc}}\in A_0(\mathcal{N}_L^{\perp})$$

and the Kiem-Li localized invariant $\int_{[\mathcal{N}_L^{\perp}]^{\mathrm{vir}}_{\mathrm{loc}}}1$ is proved to be the same as Behrend's weighted Euler characteristic $\mathrm{vw}(\mathcal{S})=\chi(\mathcal{N}_L^{\perp},\nu_N)$, see [22], [24].

The moduli space $\mathcal{N}_L^{\perp}$ here is not compact, but admits a $\mathbb{C}^*$-action whose fixed loci are compact. There are two main components of the $\mathbb{C}^*$-fixed locus $(\mathcal{N}_L^{\perp})^{\mathbb{C}^*}$. The first one $\mathscr{M}^{(1)}=\mathscr{M}_L$ corresponds to the $\mathbb{C}^*$-fixed Higgs pairs (E,ϕ) such that the Higgs fields ϕ are zero. This corresponds to the moduli space of Gieseker stable sheaves on S. This component is called the *Instanton Branch* according to [14]. In this case we have

$$\mathrm{VW}(\mathcal{S})=\int_{[(\mathscr{M}_L)^{\mathrm{vir}}]}c_{\mathrm{vd}}(E^{\bullet}_{\mathscr{M}})$$

where $E^{\bullet}_{\mathscr{M}}$ is the perfect obstruction theory on $\mathscr{M}_L$, and vd is the virtual dimension. This is Ciocan-Fontanine-Kapranov/Fantechi-Göttsche virtual signed Euler number, see [25, §3].

If the surface DM stack S satisfies $K_S\leqslant 0$, then [48] [49] prove that the only $\mathbb{C}^*$-fixed Higgs pairs must have $\phi=0$, and

$$\mathrm{VW}(\mathcal{S})=(-1)^{\mathrm{vd}}\chi(\mathscr{M}_L)$$

and this signed Euler number of $\mathscr{M}_L$ is the same as $\mathrm{vw}(\mathcal{S}) = \chi(\mathcal{N}_L^{\perp}, \nu_N) = \chi(\mathscr{M}_L, \nu_N|_{\mathscr{M}_L})$, see [22, §5].

The second component $\mathscr{M}^{(2)}$ corresponds to the $\mathbb{C}^*$-fixed Higgs pairs (E, ϕ) such that the Higgs fields ϕ are nonzero. This component is called the *Monopole Branch*. The case usually happens for general type surfaces or DM stacks. In [48, §8], [25, §4] this component is proved to be the union of nested Hilbert schemes on S. Gholampour and Thomas [11] has found a new way to calculate the invariants by reducing the integration on nested Hilbert schemes to Hilbert schemes, and it is very interesting to see what happens for surface DM stacks.

In this paper we are more interested in the invariants vw, although they are not deformation invariant. Since the invariants vw is defined by weighted Euler characteristics, it can be applied to study the geometric Eisenstein series associated with a smooth divisor in the surface S following Kapranov [34]. At least for the surface DM stacks S satisfying $K_{\mathcal{S}} < 0$, vw gives the same Vafa-Witten invariants VW which are deformation invariant. For K3 surfaces, $\mathrm{VW}(S) = \mathrm{vw}(S)$, and **[?]** prove this result for twisted Vafa-Witten invariants $\mathrm{VW}^{\mathrm{tw}}(\mathcal{S}) = \mathrm{vw}^{\mathrm{tw}}(\mathcal{S})$ for K3 gerbes $\mathcal{S}$. In literature this invariant vw is closely related to the local Donaldson-Thomas invariants and is also predicted to be modular forms, see [53].

1.2. **The moduli space of Higgs pairs on root stacks.** Root stacks provide interesting two dimensional Deligne-Mumford stacks. We provide a brief explanation here for the special case and leave the detailed definition to §2. In general it is convenient to use log schemes to define root stacks. It is very interesting to see how the Vafa-Witten invariants inspired by the S-duality conjecture can be put into the definition of general root stacks. In this paper we first deal with the case of the Vafa-Witten invariants of root stacks $\mathcal{S}$ given by a pair (S, D), where S is a smooth projective surface and $D \subset S$ a smooth divisor curve.

Let us consider the quotient stack $[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$, where $\mathbb{G}_{\mathrm{m}}$ acts on $\mathbb{A}^1$ naturally. The category (L, s) of line bundles with global section on S is equivalent to the category of morphisms

$$S \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}].$$

Let $d \in \mathbb{N}$ be a positive integer. Let $\Theta_d : [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}] \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$ be the morphism of stacks given by $x \mapsto x^d$ and $t \mapsto t^d$ where $x \in \mathbb{A}^1$ and $t \in \mathbb{G}_{\mathrm{m}}$. The line bundle $\mathcal{O}_S(D)$ and the section s_D gives a morphism

$$(\mathcal{O}_S(D), s_D) : S \to [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}].$$

Then

$$\mathcal{S} := \sqrt[d]{(S, D)} = S \times_{[\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]} [\mathbb{A}^1/\mathbb{G}_{\mathrm{m}}]$$

is the fibred product of stacks. Let $\pi : \mathcal{S} \to S$ be the projection and S is the coarse moduli space of $\mathcal{S}$. The stacky locus of $\mathcal{S}$ is exactly $\mathcal{D} := \pi^{-1}(D)$, which is a μ_d-gerbe over D.

The moduli space of stable coherent sheaves on any projective DM stack has been studied by F. Nironi in [46]. A review is given in [25, §2]. Let us look at the d-th root stack $\mathcal{S} = \sqrt[d]{(S, D)}$. Let $\mathcal{D}^{\mathrm{red}}$ be the reduced divisor on $\mathcal{S}$ and let

$$\Xi := \bigoplus_{l=0}^{d-1} \mathcal{O}_{\mathcal{S}}(l \cdot \mathcal{D}_{\mathrm{red}}).$$

Then Ξ is a generating sheaf for $\mathcal{S}$ which means that Ξ is very ample under the morphism $\pi : \mathcal{S} \to S$. Using this generating sheaf the modified Hilbert polynomial is defined by:

$$P_{\Xi}(E, m) = \chi(\mathcal{S}, E \otimes \Xi^{\vee} \otimes p^{*}\mathcal{O}_S(m)).$$

Then we can write down

$$P_{\Xi}(E, m) = \sum_{i=0}^{\dim} \alpha_{\Xi,i} \frac{m^i}{i!},$$

where $\dim = \dim(E)$ is the dimension of the sheaf E. The reduced Hilbert polynomial for pure sheaves, and we will denote it with $p_{\Xi}(E)$; is the monic polynomial with rational coefficients $\frac{P_{\Xi}(E)}{\alpha_{\Xi,d}}$ Then let E be a pure coherent sheaf, it is semistable if for every proper subsheaf $F \subset E$ we have $p_{\Xi}(F) \leqslant p_{\Xi}(E)$ and it is stable if the same is true with a strict inequality. Then fixing a Hilbert polynomial H, the moduli stack of semistable coherent sheaves $\mathscr{M} := \mathscr{M}_H$ on $\mathcal{S}$ is constructed in [46]. If the stability and semistability coincide, the coarse moduli space $\mathscr{M}$ is a projective scheme.

For the pair (S, D), Maruyama and K. Yokogawa [41] defined the notion of parabolic sheaves. A parabolic sheaf E is given by a *parabolic structure* on E which is given by a length d-filtration

$$E = F_0(E) \supset F_1(E) \supset \cdots \supset F_d(E) = E(-D),$$

together with a system of weights

$$0 \leqslant \alpha_0, \alpha_1, \cdots, \alpha_{d-1} < 1.$$

We call $E_{\bullet} = (E, F_i(E))$ a parabolic sheaf associated with the divisor D. Then Maruyama and Yokogawa defined the parabolic stability condition in Definition 3.3 on the parabolic sheaves and construct the moduli space of parabolic stable sheaves, see §3.1. It turns out that the stability using the generating sheaf Ξ on $\mathcal{S}$ is the same as the parabolic stability of Maruyama and Yokogawa, see (3.1.2). Then from [46], or more generally [51], the moduli space of stable sheaves on a root stack is isomorphic to the moduli space of parabolic stable sheaves on a pair (S, D), where D is a smooth divisor curve by fixing a suitable Hilbert polynomial, see Theorem 3.6.

The definition of Maruyama and Yokogawa on parabolic sheaves was generalized by Yokogawa [55] to define parabolic Higgs pairs, and Gieseker semistability on them. The definition is almost the same as the case of parabolic sheaves except that we take the Higgs field ϕ-invariant subsheaves of the Higgs pair (E, ϕ). The moduli space $\mathcal{N}_{\text{pa}}$ of semistable parabolic Higgs pairs with a fixed Hilbert polynomial H was also constructed. Our first main result is:

Theorem 1.1. *(Theorem 3.13) Let $\mathcal{S} = \sqrt[d]{(S, D)}$ be the root stack of S with respect to the smooth divisor D. Choosing the generating sheaf $\Xi = \oplus_{l=0}^{d-1}\mathcal{O}_{\mathcal{S}}(l\mathcal{D}_{\text{red}})$, and fixing some modified Hilbert polynomial $H \in \mathbb{Q}[m]$. Then the moduli space $\mathcal{N}_H := \mathcal{N}_H(\mathcal{S})$ of semistable Higgs pairs on the root stack $\mathcal{S}$ is isomorphic to the moduli space $\mathcal{N}_{\text{pa}} := \mathcal{N}^{H,\alpha}_{\text{pa},(S,D)}$ of semistable parabolic Higgs pairs on (S, D). Their corresponding stable open subspaces $\mathcal{N}^s_H$ and $\mathcal{N}^s_{\text{pa}}$ are also isomorphic.*

Thus one can use the moduli space of Higgs pairs on the root stack $\mathcal{S}$ to study the moduli of parabolic Higgs pairs, and more interestingly study the Vafa-Witten invariants on them.

1.3. **The relative moduli spaces.** In the work [34], Kapranov studied the relation between the generating functions of the moduli space of stable sheaves on the surface S and the geometric Eisenstein series for any Kac-Moody group. For the pair (S, D), where S is a smooth projective surface and $D \subset S$ a smooth curve. Let $S^0 := S \backslash D$. By fixing a coherent sheaf E^0 on S^0 and a Hilbert polynomial H, Kapranov constructed the moduli space $\mathscr{M}^H(S; D)$ of sheaves E on S such that $E|_{S^0} \cong E^0$. This moduli space is an ind-scheme, and if the self-intersection number D_S^2 is negative, then $\mathscr{M}^H(S; D)$ is a fine moduli space, see [34, Theorem 2.2.1]. This relative situation studied in this paper is related to the restriction of coherent sheaves on the curve $D \subset S$. A slope stable torsion free coherent sheaf E on S is still stable when restricted in sufficiently large degree smooth curves $D \subset S$, see [43], [10], [5]. In an interesting case of blow-up so that the smooth divisor is $D = \mathbb{P}^r$, a slope semistable sheaf on S, when restricted to D can be made to be optimal (close to semistable) in the paper [8]. It is interesting to study the restriction theorem for Gieseker stability and for stable Higgs sheaves, and relate the S-duality on surfaces to Langlands duality on curves.

We study the Kapranov relative moduli space using formal moduli theory as in [28]. We do this on the root stack $\mathcal{S}$ with $\mathcal{D} \subset \mathcal{S}$ an orbifold divisor. Let $R := \kappa[\![t]\!]$ the the discrete valuation ring and $K = \kappa((t))$ the field of Laurent series, which is a non-archimedean field with valuation val. For the stack $\mathcal{S}$, let $\widehat{\mathcal{S}} \to \mathrm{Spf}(R)$ be the trivial t-adic completion of $\mathcal{S}$ so that $\widehat{\mathcal{S}}$ is a stft formal R-scheme. We consider the moduli space $\mathscr{M}^H(\mathcal{S}; \mathcal{D})$ of sheaves E on $\mathcal{S}$ such that $E|_{\mathcal{S}^0} \cong E^0$ for a fixed E^0 on $\mathcal{S}^0 := \mathcal{S} \backslash \mathcal{D}$. Then we have

Theorem 1.2. *The moduli space $\mathscr{M}^H(\mathcal{S}; \mathcal{D})$ can be made to a formal subscheme of $\mathscr{M}_R^H(\widehat{\mathcal{S}})$, where $\mathscr{M}_R^H(\widehat{\mathcal{S}})$ is the formal moduli scheme of stable sheaves E on $\widehat{\mathcal{S}}$ with Hilbert polynomial H.*

Let $\widehat{\mathcal{S}}_{\mathcal{D}}$ be the formal completion of $\mathcal{S}$ along the closed substack $\mathcal{D}$. Then the similar moduli spaces and relative moduli spaces can be considered. We have:

Theorem 1.3. *(Theorem 4.1) The moduli stack $\widehat{\mathscr{M}}_R^H(\widehat{\mathcal{S}}_{\mathcal{D}}; \mathcal{D})$ of stable relative sheaves E on $\widehat{\mathcal{S}}_{\mathcal{D}}$ with Hilbert polynomial H and $E|_{\mathcal{S}^0_{\mathcal{D}}} \cong E^0$ is a subformal moduli stack of $\widehat{\mathscr{M}}_R(\widehat{\mathcal{S}}_{\mathcal{D}}, H)$.*

The formal moduli scheme and formal relative parabolic scheme of stable sheaves on $\widehat{S}$ can be similarly defined. We also define and construct the formal moduli space of stable parabolic or stable Higgs pairs (E, ϕ) on the formal scheme $\widehat{S}$ or $\widehat{\mathcal{S}}$. The result is:

Let $\mathcal{T}_m := \mathrm{Spec}(R/(t^{m+1}))$, where t is the uniformizer of the discrete valuation ring R. Then we have:

Theorem 1.4. *(Theorem 4.6) Let $\widehat{S}$ be a stft formal scheme over R, and let*

$$\widehat{S}_m := \widehat{S} \times_R \mathcal{T}_m.$$

Then

$$\varinjlim_m \mathcal{N}^H_{\mathcal{T}_m,\mathrm{pa}}(\widehat{S}_m) \cong \mathcal{N}^H_{R,\mathrm{pa}}(\widehat{S}),$$

where $\mathcal{N}^H_{R,\mathrm{pa}}(\widehat{S})$ is the formal moduli stack of stable parabolic formal sheaves over $\widehat{S}$ with Hilbert polynomial H.

It is interesting to point out that in general the relative moduli space $\mathscr{M}^H(S,D)$ is not the same the moduli space of stable coherent sheaves $\mathscr{M}^H(D)$ on the smooth curve D. It is of course interesting to study the relation between these two moduli spaces, and one hopes that the S-duality on the surface S can be reduced to the Langlands duality on the curve D, see [31] for a progress in this direction.

1.4. **Vafa-Witten invariants and the Blow-up formula.** For the root stack $\mathcal{S} = \sqrt[d]{(S,D)}$, we use a K-group class $\mathbf{c} \in K_0(\mathcal{S})$ to fix a Hilbert polynomial H. Let $\mathcal{N}^{\mathbf{c}}(\mathcal{S}) := \mathcal{N}^{\perp}_{L,\mathbf{c}}(\mathcal{S})$ be the moduli space of Higgs pairs (E,ϕ) on $\mathcal{S}$ with K-group class $\mathbf{c}$ and determinant $L \in \mathrm{Pic}(\mathcal{S})$. Then the invariant

$$\mathrm{vw}_{\mathbf{c}}(\mathcal{S}) = \chi(\mathcal{N}^{\mathbf{c}}(\mathcal{S}), \nu_{\mathcal{N}})$$

is the weighted Euler characteristic weighted by the Behrend function $\nu_{\mathcal{N}} : \mathcal{N}^{\mathbf{c}}(\mathcal{S}) \to \mathbb{Z}$. The formal relative moduli space $\widehat{\mathcal{N}}^{\mathbf{c}}(\mathcal{S};\mathcal{D})$ of stable Higgs pairs such that $E|_{\mathcal{S}^0} \cong E^0$ is a subformal moduli space of the the formal completion of $\widehat{\mathcal{N}^{\mathbf{c}}(\mathcal{S})}$ of $\mathcal{N}^{\mathbf{c}}(\mathcal{S})$. The Behrend function

$$\nu_{\widehat{\mathcal{N}}} : \widehat{\mathcal{N}^{\mathbf{c}}(\mathcal{S})} \to \mathbb{Z}$$

on the formal moduli space $\widehat{\mathcal{N}^{\mathbf{c}}(\mathcal{S})}$ is defined as follows. The moduli space $\mathcal{N}^{\mathbf{c}}(\mathcal{S})$ is isomorphic to the moduli space $\mathcal{N}^{\mathrm{tor}}$ of stable torsion sheaves $\mathcal{E}_\phi$ (associated with a Higgs pair (E,ϕ)) on the Calabi-Yau DM stack $\mathcal{X} = \mathrm{Tot}(K_{\mathcal{S}})$. The moduli space $\mathcal{N}^{\mathrm{tor}}$ is a d-critical scheme in the sense of Joyce [33], i.e., locally the critical locus of a regular function on a higher dimension smooth scheme. Then the Behrend function $\nu_{\mathcal{N}}$ is the Euler characteristic of the Milnor number, see [2, §2]. After taking the completion, the Behrend function $\nu_{\widehat{\mathcal{N}}}$ on the formal space $\widehat{\mathcal{N}^{\mathbf{c}}(\mathcal{S})}$ is the same as the Behrend function on the formal space $\widehat{\mathcal{N}^{\mathrm{tor}}}$ since both of them have the same Milnor number, see [23].

Then we define the relative invariant by:

$$\mathrm{vw}_{\mathbf{c}}(\mathcal{S};\mathcal{D}) = \chi(\widehat{\mathcal{N}}^{\mathbf{c}}(\mathcal{S};\mathcal{D}), \nu_{\widehat{\mathcal{N}}}|_{\widehat{\mathcal{N}}^{\mathbf{c}}(\mathcal{S};\mathcal{D})}). \tag{1.4.1}$$

All of above definitions and constructions work for the pair (S,D), and we have the relative invariant

$$\mathrm{vw}_{\mathbf{c}}(S;D) = \chi(\widehat{\mathcal{N}}^{\mathbf{c}}(S;D), \nu_{\widehat{\mathcal{N}}}|_{\widehat{\mathcal{N}}^{\mathbf{c}}(S;D)}). \tag{1.4.2}$$

Let us fix to the case (S,D) such that $D = \mathbb{P}^1$ and $D_S^2 = -\deg$ is negative. The relative moduli space in this case $\mathscr{M}(S;D)$ of stabel relative coheretn sheaves with K-class $\mathbf{c}$ is a fine moduli space according to Kapranov [34, Theorem 2.2.1]. Then the relative moduli space of Higgs pairs $\mathcal{N}^{\mathbf{c}}(S;D)$ is also a fine moduli space and we can take $\widehat{\mathcal{N}}^{\mathbf{c}}(S;D)$ as the trivial formal completion over $\mathrm{Spf}(R)$. The K-class in this case can be chosen as $\mathbf{c} = (\mathrm{rk}, c_1, c_2) \in H^*(S)$, where rk is the rank of the stable sheaves E on S, c_1, c_2 are the first and second Chern class of E.

On the other hand, let $\widehat{S}_D$ be the formal completion of S along D, then it is a stft formal scheme over $\mathrm{Spf}(R)$ with the underlying support scheme D. The formal scheme $\widehat{S}_D$ can be understood as the formal thickening of D inside S or the formal neighborhood of D inside the normal bundle $N_{D/S}$. By relative GAGA in [16, Theorem 9.2.1], if $j : \widehat{S}_D \to S$ is the canonical morphism as formal schemes, then the formal completion of the étale cohomology of coherent sheaf E on S is isomorphic to the the étale cohomology of the formal completion $\widehat{E}$ on $\widehat{S}_D$. We can define the Hilbert polynomials of $\widehat{S}_D$, hence the Gieseker stability. Let $\mathscr{M}_R^{\mathbf{c}}(\widehat{S}_D; D)$ be the formal moduli scheme of relative stable coherent sheaves E on $\widehat{S}_D$ such that $E|_{\widehat{S}_D^0} \cong E^0$; and $\mathcal{N}_R^{\mathbf{c}}(\widehat{S}_D; D)$ be the formal moduli scheme of relative Higgs pairs (E, ϕ) on $\widehat{S}_D$ such that $E|_{\widehat{S}_D^0} \cong E^0$. We define the formal geometric Eisenstein series as:

Definition 1.5. *We define:*

$$E^{\mathrm{for}}(q) = E_{\widehat{S}_D, \widehat{E}^0}(q) = \sum_{\mathbf{c} \in K_0(\widehat{S}_D)} \chi(\widehat{\mathscr{M}_R^{\mathbf{c}}}(\widehat{S}_D; D)) q^{\mathbf{c}}$$

and

$$F^{\mathrm{for}}(q) = F_{\widehat{S}_D, \widehat{E}^0}(q) = \sum_{\mathbf{c} \in K_0(\widehat{S}_D)} \chi(\widehat{\mathcal{N}_R^{\mathbf{c}}}(\widehat{S}_D; D)) q^{\mathbf{c}}$$

where for the formal schemes, the Euler characteristic $\chi(\widehat{\mathscr{M}_R^{\mathbf{c}}}(\widehat{S}_D; D))$ and $\chi(\widehat{\mathcal{N}_R^{\mathbf{c}}}(\widehat{S}_D; D))$ are defined using étale cohomology.

If the Vafa-Witten invariants $\mathrm{VW} = \mathrm{vw}$ which is the case when $K_S < 0$, we have a blow-up formula on [40].

Theorem 1.6. *(Theorem 5.13) Let $\sigma : \widetilde{S} \to S$ be the blow-up of the surface S along a point $P \in S$. Let*

$$\mathrm{vw}^{\widetilde{S}}_{\widetilde{c}_1, \widetilde{c}_2} = \chi(\mathcal{N}_L^{\perp}(\widetilde{S}), \nu_{\mathcal{N}_L^{\perp}}), \quad \mathrm{vw}^{S}_{c_1, c_2} = \chi(\mathcal{N}_L^{\perp}(S), \nu_{\mathcal{N}_L^{\perp}})$$

be the small Vafa-Witten invariants of $\widetilde{S}, S$ respectively with topological data $\widetilde{c}_1, c_1, \widetilde{c}_2 = c_2$. Let $\mathscr{M}_L(\widetilde{S})$ and $\mathscr{M}_L(S)$ be the moduli spaces of stable torsion free sheaves on $\widetilde{S}, S$ respectively with topological data $\widetilde{c}_1, c_1, \widetilde{c}_2 = c_2$. Assume that

$$\mathrm{vw}^{\widetilde{S}}_{\widetilde{c}_1, \widetilde{c}_2} = (-1)^{\mathrm{vd}} \chi(\mathscr{M}_L(\widetilde{S})), \quad \mathrm{vw}^{S}_{c_1, c_2} = (-1)^{\mathrm{vd}} \chi(\mathscr{M}_L(S)).$$

Then we have:

$$\sum_n \mathrm{vw}^{\widetilde{S}}_{\widetilde{c}_1, n} q^n = \frac{E^{\mathrm{for}}_{\widehat{S}_D, \widehat{E}^0}(q)}{\left(\prod_{n \neq 0}(1-q^n)\right)^2} \cdot \sum_n \mathrm{vw}^{S}_{c_1, n} q^n.$$

The series $E_{\widehat{S}_D, \widehat{E}^0}(q)$ can be calculated very explicitly, see Proposition 5.11.

Remark 1.7. *Theorem 1.6 makes sense for projective surfaces S with $K_S < 0$ or K3 surfaces, since the $\mathbb{C}^*$ fixed loci on $\mathcal{N}_L^{\perp}(\widetilde{S})$ and $\mathcal{N}_L^{\perp}(S)$ are $\mathscr{M}_L(\widetilde{S})$ and $\mathscr{M}_L(S)$, which are smooth. These are the Instanton branches of the fixed loci. For smooth projective surfaces S with $K_S > 0$, for instance smooth general type surfaces, it is interesting to study the blow-up formula for such type of surfaces. The $\mathbb{C}^*$ fixed part for $\mathcal{N}_L^{\perp}(S)$ contains the second component $\mathscr{M}^{(2)}$ such that for $(E, \phi) \in \mathscr{M}^{(2)}$, $\phi \neq 0$. The component $\mathscr{M}^{(2)}$ is* *called the Monopole branch for the fixed loci, which is given by nested Hilbert schemes, see* [48]*. One hopes that for the fixed curve $D \subset S$, the Hitchin fibration map for D can help for the blow-up formula.*

1.5. **Discussion of the stacky blow-up.** We also give a explanation of the blow-up on in the stacky case. Let $\mathcal{S}$ be a projective surface with one isolated quotient sigularity, for instance the quintic surface with only one point P which is the type of A_2 singularity, and let $\pi : \mathcal{S} \to S$ be the morphism to its coarse moduli space. Let $\sigma : \widetilde{S} \to S$ be blow-up along the singular point $P \in S$. We put this into a diagram:

$$\begin{array}{ccc} \widetilde{S} & & \mathcal{S} \\ & \searrow^{\sigma} \quad \swarrow^{\pi} & \\ & S & \end{array}$$

In general it is interesting to compare the Vafa-Witten invariants of $\widetilde{S}$ and the Vafa-Witten invariants of $\mathcal{S}$. Note that $K_0(\widetilde{S}) \cong K_0(\mathcal{S})$ have the same K-groups. Fixing a K-group class $\mathbf{c} = (r, c_1, c_2) \in H^*(\widetilde{S})$. If the surface $\widetilde{S}$ and $\mathcal{S}$ all satisfy $K_{\widetilde{S}} < 0, K_{\mathcal{S}} < 0$ or are K3 surfaces, then

$$\mathrm{VW}_{\mathbf{c}}(\widetilde{S}) = \mathrm{vw}_{\mathbf{c}}(\widetilde{S}) = (-1)^{\mathrm{vd}}\chi(\mathscr{M}^{\mathbf{c}}(\widetilde{S}))$$

where $\mathrm{vd} = 2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(\mathcal{O}_{\widetilde{S}})$. And for such a K-group class $\mathbf{c} \in K_0(\mathcal{S})$, the invariant

$$\mathrm{VW}_{\mathbf{c}}(\mathcal{S}) = \mathrm{vw}_{\mathbf{c}}(\mathcal{S}) = (-1)^{\mathrm{vd}}\chi(\mathscr{M}^{\mathbf{c}}(\mathcal{S}))$$

where vd is the virtual dimension of the moduli space $\mathscr{M}^{\mathbf{c}}(\mathcal{S})$. We hope that the invariants vw of $\widetilde{S}$ and $\mathcal{S}$ are related by wall crossings.

The series $E_{\widehat{S}_D, \widehat{E}^0}(q)$ can also be calculated very explicitly,

Proposition 1.8. ([34, Theorem 7.4.6]) *In the case of stacky blow-up such that $D = \mathbb{P}^1$, and $D_S^2 = -2$, we have*

$$E_{\widehat{S}_D, \widehat{E}^0}(q) = \sum_{a \in \mathbb{Z}} q^{-\Psi(a,a)} = \sum_{a \in \mathbb{Z}} q^{-a^2}.$$

1.6. **Outline.** The paper is organized as follows. We review the construction of root stacks in §2, where in §2.1 we collect some materials on log schemes; and in §2.2 we define root stacks on log schemes. §3 studies the moduli space of stable Higgs pairs on the rot stack $\mathcal{S}$ and parabolic stable Higgs pairs on the pair (S, D) for $D \subset S$ a smooth divisor. More precisely, in §3.1 we review the construction of the moduli of parabolic stable sheaves on (S, D); and in §3.2 we prove that the moduli space of parabolic Higgs pairs on (S, D) is naturally isomorphic to the moduli space of stable Higgs pairs on the root stack $\mathcal{S}$. The moduli space of relative stable sheaves and Higgs pairs on the root stack $(\mathcal{S}, \mathcal{D})$ and on the projective surface (S, D) are constructed in §4, where in §4.1 the construction of Kapranov [34] is reviewed; and in §4.2 we construct the formal relative space of stable Higgs sheaves on $(\mathcal{S}, \mathcal{D})$ and (S, D). Finally in §5 we put the Vafa-Witten invariants into the picture of relative moduli of Higgs sheaves. In more detail in §5.1 we review the geometric Eisenstein series; in §5.2 we talk about the moduli space of Higgs bundles on curves; and finally in §5.3 we study the case that $D = \mathbb{P}^1$ and get the blow-up formula for the Vafa-Witten invariants in the special case.

1.7. **Convention.** We work over an algebraically closed field κ with characteristic zero except in some sections we will make clear over characteristic p. We use $\mathbb{C}^*$ to represent the multiplicative group $\mathbb{A}^1_\kappa \backslash \{0\}$. All the moduli spaces of stable sheaves and Higgs pairs are over κ.

In §4 we use formal moduli spaces over a discrete valuation ring as in [28]. Let us fix some notations. For the formal schemes, let R be a complete discrete valuation ring $R = \kappa[\![t]\!]$, with quotient field $\mathbb{K} := \kappa((t))$, and perfect residue field κ. We fix a uniformizing element t in R, i.e., a generator of the maximal ideal. The field $\mathbb{K}$ is a non-archimedean field with valuation v such that $v(t) = 1$. The absolute value $|\cdot| = e^{-v(\cdot)}$. All formal schemes over R is *stft* (separate and topologically of finite type) in sense of [45], and the non-archimedean analytic spaces over $\mathbb{K}$ are quasi-compact Berkovich analytic spaces [4].

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2. ROOT STACKS

In this section we recall the knowledge of root stacks. Our main references are [6], [51],[7]. Let d be a positive integer, the d-th root stack associated with a scheme S and a Cartier divisor D was constructed in [7]. The notion is generalized to log schemes by [50], [51].

2.1. **Log schemes.** Let S be a scheme and we denoted by Div_S the fibred category over the small étale site $S_{\text{ét}}$ consisting of pairs (L, s) where L is an invertible sheaf and s is a global section.

Definition 2.1. *A* Deligne-Faltings *(DF) structure on S is a symmetric monoidal functor $L : A \to \mathrm{Div}_S$ with trivial kernel, where A is a sheaf of monoids on the small étale site $S_{\text{ét}}$.*

A logarithmic scheme *is a scheme S equipped with a DF structure.*

Remark 2.2. (1) *That the trivial kernel for $L : A \to \mathrm{Div}_S$ means that for any $U \to S$ étale, we get a morphism $A(U) \to \mathrm{Div}_{S(U)}$, and the only section of $A(U)$ that goes to $(\mathcal{O}_U, 1)$ is the zero section.*

(2) *We denote the log scheme by (S, A, L), or just S if we understand the data (A, L).*

(3) *Let us recall the standard definition of a log scheme, see* [**?**]. *A log scheme is a pair (S, M) such that M is a sheaf of monoids on $S_{\text{ét}}$, and there is a morphism*

$$\alpha : M \to \mathcal{O}_S$$

such that $\alpha|_{\alpha^{-1}(\mathcal{O}_S^\times)} : \alpha^{-1}(\mathcal{O}_S^\times) \xrightarrow{\sim} \mathcal{O}_S^\times$ is an isomorphism. A log scheme is quasi-integral if the natural resulting action of $\mathcal{O}_S^\times$ on M is free. Let us relate this *to the Definition 2.1. Given a morphism of sheaves of monoids*

$$\alpha : M \to \mathcal{O}_S$$

one takes the stack quotient by $\mathcal{O}_S^\times$ and obtain a symmetric monoidal functor

$$L : \overline{M} \to [\mathcal{O}_S/\mathcal{O}_S^\times] \cong \mathrm{Div}_S .$$

Let $A = \overline{M}$. This means that a section of A is send by L to the dual L_a of the invertible sheaf associated to the $\mathbb{G}_{\mathrm{m}}$-torsor given by the fiber $M_a \to \mathcal{O}_S$ gives the section of L_a. On the other hand, if $L : A \to \mathrm{Div}_S$ is a DF-structure, we take the fibred product $A \times_{\mathrm{Div}_S} \mathcal{O}_S \to \mathcal{O}_S$, and verify that $M = A \times_{\mathrm{Div}_S} \mathcal{O}_S$ is equivalent to a sheaf.

Definition 2.3. *A morphism of log schemes $(S, A, L) \to (T, B, N)$ is a pair $(f, f^\flat)$, where $f : S \to T$ is a morphism of schemes, and $f^\flat : f^*(B, N) \to (A, L)$ is a morphism of DF structures of S. Here $f^*(B, N) = (f^*B, f^*N)$ is a pullback of DF structure, see* [6], *and the morphism*

$$f^\flat : f^*(B, N) \to (A, L)$$

of DF structures is given by a pair $f^\flat = (f^\flat, \alpha)$ such that

$$f^\flat : f^*B \to A$$

is a morphism as sheaves of monoids, and $\alpha : L \circ f^\flat \cong f^*N$ is a natural isomorphism of symmetric monoidal functors $f^*B \to \mathrm{Div}_S$.

Definition 2.4. *A morphism of log schemes*

$$(f, f^\flat) : (S, A, L) \to (T, B, N)$$

is strict *if $f^\flat$ is an isomorphism.*

We introduce the charts for the log schemes.

Definition 2.5. *A chart for a sheaf of monoids A on $S_{\acute{e}t}$ is a homomorphism of monoids*

$$\phi : P \to A(S)$$

such that the induced map of sheaves $\phi : P_S \to A$ is a cokernel, which means that the induced homomorphism between the stalks are all cokernels (the induced morphism $P/\phi^{-1}(0) \to A$ is isomorphism at each stalk.)

A sheaf of monoids A on $S_{\acute{e}t}$ is coherent if A has charts with finitely generated monoids locally for the étale topology of S. A log scheme (S, A, L) is coherent if A is coherent.

Remark 2.6. *A chart for a DF structure (A, L) on S can also be seen as a symmetric monoidal functor $P \to \mathrm{Div}_S$ for a monoid P, that indues the functor $L : A \to \mathrm{Div}_S$ by sheafifying and trivializing the kernel.*

Let us recall the Kato chart for a log scheme, which is a morphism of monoids

$$P \to \mathcal{O}_S(S)$$

that induces the log structure $\alpha : M \to \mathcal{O}_S$. Every Kato chart $P \to \mathcal{O}_S(S)$ induces a chart by composing with $\mathcal{O}_S(S) \to \mathrm{Div}(S)$.

Definition 2.7. *A* Kato chart *for a morphism of log schemes*

$$(f, f^\flat) : (S, A, L) \to (T, B, N)$$

is a chart such that the functors

$$P \to \mathrm{Div}(T), \quad Q \to \mathrm{Div}(S)$$

lift to $P \to \mathcal{O}_T(T)$ *and* $Q \to \mathcal{O}_S(S)$. *Or equivalently a Kato chart can be seen as a commutative diagram of log schemes:*

$$\begin{array}{ccc} (S, A, L) & \longrightarrow & \mathrm{Spec}(\kappa[Q]) \\ \downarrow & & \downarrow \\ (T, B, N) & \longrightarrow & \mathrm{Spec}(\kappa[P]) \end{array}$$

and a chart can be seen as a similar commutative diagram with the quotient stacks $[\mathrm{Spec}(\kappa[P])/\widehat{P}]$ *and* $[\mathrm{Spec}(\kappa[Q])/\widehat{Q}]$ *replacing* $\mathrm{Spec}(\kappa[P])$ *and* $\mathrm{Spec}(\kappa[Q])$ *respectively, where* $\widehat{P}, \widehat{Q}$ *are the Cartier duals of* $P^{\mathrm{gp}}, Q^{\mathrm{gp}}$ *respectively.*

Definition 2.8. *A log scheme* (S, A, L) *is* fine *if it is coherent and the stalks of A are fine monoids (integral and finitely generated). A log scheme* (S, A, L) *is* fine *and* saturated *(fs) if it is fine and the stalks of A are fine and saturated monoids.*

We are more interested in the following root stacks in this paper. Let S be a scheme and $D \subset S$ an effective Cartier divisor. The natural log structure associated with D is given by

$$M = \left\{ f \in \mathcal{O}_S | f|_{S \setminus D} \in \mathcal{O}^\times_{S \setminus D} \right\}$$

as a sheaf of $\mathcal{O}_S$, and the inclusion

$$\alpha : M \to \mathcal{O}_S.$$

This is called the divisorial log structure associated with D on S. If D is a simple normal crossing divisor and has components $D_1, \cdots, D_n$, then the corresponding DF structure admits a global chart:

$$\mathbb{N}^n \to \mathrm{Div}(S)$$

sending

$$e_i \mapsto (\mathcal{O}_S(Di), s_i)$$

where s_i is the canonical section of $\mathcal{O}_S(D_i)$. If D is normal crossing but not simple normal crossing, then the divisorial log structure is still *fs*, but does not admit a global chart.

2.2. **Root stacks.** Root stacks are stacks that parametrize roots of the log structure of a log scheme (S, A, L) with respect to a system of denominators $A \to B$.

Definition 2.9. *A system of denominators on a* fs *log scheme* (S, A, L) *is a coherent sheaf of monoids B on* $S_{ét}$ *with a Kummer morphism of sheaves of monoids* $j : A \to B$ *(i.e., the induced morphism between the stalks is Kummer at every point).*

Remark 2.10. *Let us recall that a homomorphism of monoids* $A \to B$ *is Kummer if it is injective, and for any* $q \in B$ *there exists a positive integer n such that nq is in the image.*

For instance the inclusion $A \to \frac{1}{d}A$ where $\frac{1}{d}A$ is the sheaf with sections $\frac{a}{d}$ where a is a section of A. If $r|m$, then we have $\frac{1}{d}A \subseteq \frac{1}{m}A$. The direct limit

$$\varinjlim_{r} \frac{1}{d}A$$

will be denoted by $A_{\mathbb{Q}}$. A section of $A_{\mathbb{Q}}$ is locally (globally if S is quasi-compact) of the form $\frac{a}{d}$ for some positive integer $d \in \mathbb{N}$. The inclusion $A \subseteq A_{\mathbb{Q}}$ is the maximal Kummer extension of A.

Definition 2.11. *Let B be a system of denominators on a log scheme (S, A, L). The root stack $\sqrt[B]{S}$ is the fibred category over the category of schemes (Sch_S) which is defined as follows:*

$$\sqrt[B]{S} : (\mathrm{Sch}_S) \to \mathit{Groupoids}$$

such that for any scheme $t : T \to S$, the category $\sqrt[B]{S}(T)$ is the category of pairs (N, α) where N is a DF structure $N : t^*B \to \mathrm{Div}_T$ over T, and α is a natural isomorphism between the pullback t^*L and the composition $t^*A \to t^*B \xrightarrow{N} \mathrm{Div}_T$, and the arrows are morphisms of DF structures compatible with the natural isomorphisms.

Moreover, if we take $B = \frac{1}{d}A$, we obtain the d-th root stack of S associated with the log structure (A, L) and we denote it by $\sqrt[d]{(S, A, L)}$.

From [6], [51], there is a natural projection

$$\pi : \sqrt[B]{S} \to S$$

to its coarse moduli space S and $\sqrt[B]{S}$ is a Deligne-Mumford stack.

2.2.1. *Charts.* If $j_0 : P \to Q$ is a Kummer morphism giving a chart for the system of denominators, then we can give a definition of root stacks using P and Q, which we denote them by $\sqrt[P]{S}$ and $\sqrt[Q]{S}$ respectively. The chart given by P corresponds to a morphism

$$S \to [\mathrm{Spec}(\kappa[P])/\widehat{P}]$$

where $\widehat{P} = D(P^{\mathrm{gp}})$ is the diagonalizable group scheme associated with P^{gp}, and the morphism $P \to Q$ induces a morphism of quotient stacks

$$[\mathrm{Spec}(\kappa[Q])/\widehat{Q}] \to [\mathrm{Spec}(\kappa[P])/\widehat{P}].$$

Then we have:

$$\sqrt[Q]{S} \cong S \times_{[\mathrm{Spec}(\kappa[P])/\widehat{P}]} [\mathrm{Spec}(\kappa[Q])/\widehat{Q}]$$

and we also have the following Cartesian diagram:

$$\begin{array}{ccccc} \sqrt[Q]{S} & \longrightarrow & [\mathrm{Spec}(\kappa[Q])/\mu_{Q,P}] & \longrightarrow & [\mathrm{Spec}(\kappa[Q])/\widehat{Q}] \\ \downarrow & & \downarrow & & \downarrow \\ S & \longrightarrow & \mathrm{Spec}(\kappa[P]) & \longrightarrow & [\mathrm{Spec}(\kappa[P])/\widehat{P}] \end{array}$$

if $P \to \mathrm{Div}(S)$ comes from a Kato chart $P \to \mathcal{O}_S(S)$, where $\mu_{Q,P}$ is the Cartier dual $D(C)$ of the cokernel C of the morphism $P^{\mathrm{gp}} \to Q^{\mathrm{gp}}$, and is a finite abelian group. The chart in this case is

$$\sqrt[Q]{S} \cong S \times_{\mathrm{Spec}(\kappa[P])} [\mathrm{Spec}(\kappa[Q])/\mu_{Q,P}].$$

2.2.2. *Root stack associated with divisorial log schemes.* Now let us do the case for the divisorial log scheme (S, D) with D simple normal crossing. In this case we understand that the stack $[\mathcal{O}_S/\mathcal{O}_S^\times]$ is isomorphic to Div_S, and Div_S classifies all the morphisms

$$S \to [\mathbb{A}^1/\mathbb{G}_m]$$

where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by multiplication, see [47, Example 5.13].

We have a global chart:

$$P = \mathbb{N}^n \to \mathrm{Div}(S).$$

Then if we have a Kummer morphism $P \to Q$, we have:

$$\sqrt[Q]{S} \cong S \times_{[\mathrm{Spec}(\kappa[P])/\widehat{P}]} [\mathrm{Spec}(\kappa[Q])/\widehat{Q}]$$

We make clear for the case that D is a smooth curve in S. Then the line bundle $(\mathcal{O}_S(D), s_D)$ defines a morphism

$$S \to [\mathbb{A}^1/\mathbb{G}_m].$$

Let $\Theta_d : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ be the morphism of stacks given by the morphism

$$x \in \mathbb{A}^1 \mapsto x^d \in \mathbb{A}^1; \quad t \in \mathbb{G}_m \mapsto t^d \in \mathbb{G}_m,$$

which sends $(\mathcal{O}_S(D), s_D)$ to $(\mathcal{O}_S(D)^{\otimes r}, s_D^d)$.

Definition 2.12. *(*[7]*) Let $\mathcal{S} := \sqrt[d]{(S, D)}$ be the stack obtained by the fibre product*

$$\begin{array}{ccc} \sqrt[d]{(S,D)} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \pi \downarrow & & \downarrow \Theta_r \\ S & \xrightarrow{(\mathcal{O}_S(D), s_D)} & [\mathbb{A}^1/\mathbb{G}_m]. \end{array}$$

We call $\mathcal{S} = \sqrt[d]{(S, D)}$ the root stack obtained from X by the r-th root construction.

Remark 2.13. *$\mathcal{S} = \sqrt[d]{(S, D)}$ is a smooth DM stack with stacky locus $\mathcal{D} := \pi^{-1}(D)$, and $\mathcal{D} \to D$ is a μ_r-gerbe over X coming from the line bundle $\mathcal{O}_S(D)|_D$. For example, the weighted projective stack $\mathbb{P}(1, a, a)$ is a root stack by taking the a-th root construction on $\mathbb{P}^2$ with divisor $\mathbb{P}^1 \subset \mathbb{P}^2$. It is pretty interesting to study how the infinite root stack can be put into the definition or calculation of Vafa-Witten invariants.*

Remark 2.14. *(Infinite root stacks) The infinite root stack is a limit version of finite root stacks, see* [50]*. We recall the infinite root stack $\sqrt[\infty]{S}$ coming from the limit version of the r-th root stacks.*

If $A \to B$ is a fixed system of denominators for a log scheme (S, A, L). Consider the subsystems

$$\left\{ A \to \frac{1}{d} B \right\}_{d \in \mathbb{N}}.$$

The limit $\varprojlim_d \sqrt[\frac{1}{d}B]{S} = \sqrt[\infty]{S}$ is the infinite root stack.

3. MODULI STACK OF STABLE SHEAVES AND STABLE PARABOLIC HIGGS SHEAVES

3.1. **Moduli space of parabolic stable sheaves.** Let (S, D) be a pair such that S is a smooth projective surface, $D \subset S$ is a smooth Cartier divisor. Usually we can let D be a Cartier divisor with normal crossings, then (S, D) naturally defines a log scheme structure on S. The moduli space of stable parabolic sheaves is also proved to isomorphic to the moduli space of stable sheaves on root stacks of S, see [6]. In this paper we don't use log schemes and only assume that X is a smooth connected projective curve in S, and we leave the general case to a future project.

Definition 3.1. *([41]) Let E be a torsion-free coherent sheaf on S. A* parabolic structure *on E is given by a length d-filtration*

$$E = F_0(E) \supset F_1(E) \supset \cdots \supset F_d(E) = E(-D),$$

together with a system of weights

$$0 \leqslant \alpha_0, \alpha_1, \cdots, \alpha_{d-1} < 1.$$

We call $E_\bullet = (E, F_i(E))$ a parabolic sheaf associated with the divisor D.

Let $G_i(E) = F_i(E)/F_{i+1}(E)$. The Hilbert polynomial $\chi(G_i(E)(m))$ is called the i-th multiplicity polynomial of the weight α_i.

Definition 3.2. *A parabolic sheaf $F_\bullet$ is a parabolic subsheaf of $E_\bullet$ if the following conditions are satisfied:*

(1) F is a subsheaf of E and E/F is torsion free.
(2) $F_i \subset E_i$ for all i.
(3) If $F_i \subseteq E_j$ for some $j > i$, then $F_i = F_j$.

3.1.1. *Parabolic stability and the moduli scheme.* As in [41, Definition 1.8], the parabolic Euler characteristic pa $-\chi(E_\bullet)$ of $E_\bullet$ is defined as:

$$\chi(E(-X)) + \sum_{i=0}^{d-1} \alpha_i \chi(G_i). \tag{3.1.1}$$

The polynomial pa $-\chi(E_\bullet(m))$ is called the parabolic Hilbert polynomial of $E_\bullet$ and the polynomial pa $-\chi(E_\bullet(m))/\operatorname{rk}(E)$ is denoted by pa $-p_{E_\bullet}(m)$.

Definition 3.3. *The parabolic sheaf of $E_\bullet$ is said to be parabolic Gieseker stable (resp. parabolic)if for every parabolic subsheaf $F_\bullet$ of $E_\bullet$ with*

$$0 < \operatorname{rk}(F) < \operatorname{rk}(E)$$

we have

$$\mathrm{pa}-p_{F_\bullet}(m) < \mathrm{pa}-p_{E_\bullet}(m), \quad (\textit{resp.}\ \mathrm{pa}-p_{F_\bullet}(m) \leqslant \mathrm{pa}-p_{E_\bullet}(m)).$$

The moduli space $\mathscr{M}_{\mathrm{pa}} := \mathscr{M}^{\mathbf{H},\alpha}_{\mathrm{pa},(S,D)}$ of Gieseker semistable parabolic sheaves is defined as follows. Fix Hilbert polynomials $\mathbf{H} = (H, H_1, \cdots, H_d)$ and $\alpha = (\alpha_1, \cdots, \alpha_l), 0 \leqslant \alpha_1 < \cdots < \alpha_d < 1$, the moduli functor

$$\mathcal{M}^{\mathbf{H},\alpha}_{\mathrm{pa},(S,D)} : \mathrm{Sch}/\kappa \to \mathit{Sets}$$

by

$$T \mapsto / \cong .$$

where (#) is the following property for the family of parabolic sheaves $E_\bullet$ on S_T.

(#): For every geometric point $t \in T$, $(E_{\bullet,t}, \alpha)$ is a parabolic semistable sheaf and $\chi(E_t(m)) = H(m)$, $\chi((E_t/F_{i+1}(E)_t)(m)) = H_i(m)$, where F_i, t is the filtration

$$E_t = F_0(E)_t \supset F_1(E)_t \supset \cdots \supset F_d(E)_t = E_t(-D);$$

The equivalence relation $\cong$ is given by:

$$E_\bullet \cong E'_\bullet$$

if and only if there exist filtrations

$$E = E^0 \supset E^1 \supset \cdots \supset E^m = 0$$
$$E' = E'^0 \supset E'^1 \supset \cdots \supset E'^m = 0$$

such that

(1) for every geometric point $t \in T$ their restriction to D_t provide with Jordan-Hölder filtrations of $E_{\bullet,t}$ and $E'_{\bullet,t}$, respectively;
(2) $\mathrm{gr}(E_\bullet) = \oplus_{i=1}^m (E^i/E^{i+1})_\bullet$ is T-flat;
(3) $\mathrm{gr}(E_\bullet) \cong \mathrm{gr}(E'_\bullet) \otimes_T L$ for some invertible sheaf L on T.

Then from [41, Theorem 3.6], the moduli functor $\mathcal{M}^{\mathbf{H},\alpha}_{\mathrm{pa},(S,D)}$ is represented by a quasi-projective scheme $\mathscr{M}_{\mathrm{pa}} := \mathscr{M}^{\mathbf{H},\alpha}_{\mathrm{pa},(S,D)}$ if $(\mathbf{H}, \alpha)$ is bounded. The stable part $\mathscr{M}^s_{\mathrm{pa}} \subset \mathscr{M}_{\mathrm{pa}}$ is an open subscheme.

3.1.2. *Moduli space of stable sheaves on root stacks.* For the root stack $\mathcal{S} = \sqrt[d]{(S,D)}$, let

$$p : \mathcal{S} \to S$$

be the map to its coarse moduli space, and $\mathcal{D} := p^{-1}(D)$ be the stacky divisor on $\mathcal{S}$. The generating sheaf for $\mathcal{S}$ is chosen as

$$\Xi = \oplus_{i=0}^{d-1} \mathcal{O}_{\mathcal{S}}(\mathcal{D}^{\frac{i}{r}}).$$

Let $\mathrm{Coh}_{\mathcal{S}}$ be the abelian category of coherent sheaves on $\mathcal{S}$, and $\mathrm{Par}_{\frac{1}{d}}(S, D)$ the abelian category of parabolic sheaves on (S, D) with length d. Then [46] defines two functors:

$$\mathcal{F}_{\mathcal{S}} : \mathrm{Coh}_{\mathcal{S}} \to \mathrm{Par}_{\frac{1}{d}}(S, D)$$

by

$$E \mapsto \mathcal{F}_{\mathcal{S}}(E)$$

where $\mathcal{F}_{\mathcal{S}}(E)_l = p_*(\mathsf{E} \otimes \mathcal{O}_{\mathcal{S}}(-l\mathcal{D}))$. Another functor

$$G_{\mathcal{S}} : \mathrm{Par}_{\frac{1}{r}}(S, D) \to \mathrm{Coh}_{\mathcal{S}}$$

by

$$E_\bullet \mapsto \int^{\mathbb{Z}} g_{\mathcal{S}}(E_\bullet)(d, d)$$

where $\int^{\mathbb{Z}} g_{\mathcal{S}}(E_\bullet)(d, d)$ is the colimit of wedges:

$$\begin{array}{ccc} g_{\mathcal{S}}(E_\bullet)(d, m) & \xrightarrow{f_{d,m}} & g_{\mathcal{S}}(E_\bullet)(d, d) \\ \Big\downarrow{\scriptstyle h_{d,m}} & & \Big\downarrow{\scriptstyle w(d)} \\ g_{\mathcal{S}}(E_\bullet)(m, m) & \xrightarrow{w(m)} & \mathcal{G} \end{array}$$

where

(1) $g_{\mathcal{S}}(E_\bullet) : \mathbb{Z}^0 \times \mathbb{Z} \to \mathrm{Coh}_{\mathcal{S}}$ is a map given by:

$$(d, m) \mapsto \mathcal{O}_{\mathcal{S}}(d\mathcal{D}) \otimes p^* E_m;$$

(2) $m \geqslant l$ is an arrow in $\mathbb{Z}$, and the arrow $h_{d,m}$ is induced by the canonical section of the divisor, the arrow $f_{d,m}$ is induced by the filtration $p^* E_\bullet$, the arrow $w(r)$ is a dinatural transformation and $\mathcal{G}$ is a sheaf in $\mathrm{Coh}_{\mathcal{S}}$.

We define that a parabolic sheaf $E_\bullet \in \mathrm{Par}_{\frac{1}{d}}(S, D)$ to be torsion free if E_0 is torsion free.

Theorem 3.4. *(*[46], [6]*) The functor $G_{\mathcal{S}}$ maps torsion free sheaves on S to torsion free sheaves on $\mathcal{S}$. Moreover, $\mathcal{F}_{\mathcal{S}}$ and $G_{\mathcal{S}}$ are inverse to each other when applied to torsion free sheaves.*

Let $T \in \mathrm{Sch}_\kappa$ be a scheme, and $p_1 : S_T = S \times_\kappa T \to S$ he the natural morphism. The family of parabolic sheaves $E_\bullet \in \mathrm{Par}_{\frac{1}{d}}(S_T, p_1^* D)$ is $\mathcal{O}_T$-flat if for every $l, m (m > l)$ every cokernel $E_l \to E_m \to Q_{l,m}$ is $\mathcal{O}_T$-flat. Then from [46, Lemma 7.9], the functor $G_{\mathcal{S}}$ maps flat families of torsion free parabolic sheaves on S to flat families of torsion free sheaves on the root stack $\mathcal{S}$. The functor $\mathcal{F}_{\mathcal{S}}$ maps flat families of torsion free sheaves on $\mathcal{S}$ to flat families of torsion free parabolic sheaves on S.

We have the following result of the corresponding moduli space. First we explain a bit about the correspondence on the stability. Recall for the root stack $\mathcal{S}$, the modified Hilbert polynomial associated with the generating sheaf Ξ is defined as:

$$P_\Xi(E, m) = \chi(\mathcal{S}, E \otimes \Xi^\vee \otimes p^* \mathcal{O}_S(m)).$$

Then we can write down

$$P_\Xi(E, m) = \sum_{i=0}^{\dim} \alpha_{\Xi,i} \frac{m^i}{i!},$$

where $\dim(E)$ is the dimension of the sheaf E. The reduced Hilbert polynomial for pure sheaves, and we will denote it with $p_\Xi(E)$; is the monic polynomial with rational coefficients $\frac{P_\Xi(E)}{\alpha_{\Xi,\dim}}$.

Definition 3.5. *The sheaf E is said to be Gieseker stable (resp. semistable) if for every subsheaf $F \subseteq E$ with we have*

$$p_\Xi(F)(m) < p_\Xi(E)(m), \quad (resp. p_\Xi(F)(m) \leqslant p_\Xi(E)(m)).$$

Since $\Xi = \oplus_{l=0}^{d-1} \mathcal{O}_{\mathcal{S}}(l\mathcal{D}_{\mathrm{red}})$,

$$P_\Xi(E, m) = \sum_{i=0}^{d-1} \chi(\mathcal{S}, E \otimes \mathcal{O}_{\mathcal{S}}(-l\mathcal{D}_{\mathrm{red}}) \otimes p^* \mathcal{O}_S(m)). \tag{3.1.2}$$

This is the same as (3.1.1). Therefore the Gieseker stability of coherent sheaves on the root stack $\mathcal{S}$ is equivalent to the parabolic sheaves on (S, X). Let us fix a Hilbert polynomial H and let $\mathscr{M} := \mathscr{M}_{\mathcal{S}}^H$ the moduli space of semistable sheaves with Hilbert polynomial H.

Theorem 3.6. *(*[46, §7.2], [51]*) Let $\mathcal{S} = \sqrt[d]{(S, D)}$ be the d-th root stack corresponding to the smooth divisor D and $d \geqslant 1$. Fix one Hilbert polynomial $H \in \mathbb{Q}[m]$ and the generating sheaf Ξ for $\mathcal{S}$, the moduli space $\mathscr{M} := \mathscr{M}_{\mathcal{S}}^H$ of semistable sheaves with Hilbert polynomial*

H is isomorphic to the moduli space $\mathscr{M}_{\mathrm{pa}} := \mathscr{M}^{\mathbf{H},\alpha}_{\mathrm{pa},(S,D)}$ of semistable parabolic sheaves on (S, D). The corresponding open stable parts are also isomorphic.

3.2. **Moduli space of parabolic Higgs sheaves.**

3.2.1. *Moduli of parabolic Higgs sheaves.* We first review the moduli space of parabolic Higgs pairs by Yokogawa [55]. We still fix the pair (S, D).

Definition 3.7. *A parabolic Higgs sheaf is a pair $(E_\bullet, \phi)$, where $E_\bullet$ is a parabolic sheaf in Definition 3.1, and*

$$\phi : E_\bullet \to E_\bullet \otimes_S K_S$$

is a homomorphism which is called a parabolic Higgs field.

Remark 3.8. *The tensor product $E_\bullet \otimes K_S$ is also a parabolic sheaf with $(E_\bullet \otimes K_S)_\alpha = E_\alpha \otimes_S K_S$.*

Definition 3.9. *A parabolic subsheaf $F_\bullet \subset E_\bullet$ is called "ϕ-invariant" if for all $0 \leqslant \alpha < 1$, $\phi(F_\alpha)$ is contained in $F_\alpha \otimes K_S$.*

A $\mathcal{O}_S$-homomorphism

$$f : (E_\bullet, \phi) \to (F_\bullet, \phi')$$

is said to be a homomorphism of parabolic Higgs pairs if $\phi' \circ f = (f \otimes \mathrm{id}_{K_S}) \circ \phi$ and f is a parabolic homomorphism of $E_\bullet$ to $F_\bullet$. We call $(F_\bullet, \phi')$ a parabolic sub-Higgs pair of $(E_\bullet, \phi)$ if F is a coherent subsheaf of E, $F_\alpha \subseteq E_\alpha$ for all α and $\phi|_F = \phi'$.

Definition 3.10. *A parabolic Higgs pair $(E_\bullet, \phi)$ is said to be Gieseker stable (resp. parabolic Gieseker semistable) if for every ϕ-invariant parabolic subsheaf $F_\bullet$ of $E_\bullet$ with $0 \neq F \neq E$, we have*

$$\mathrm{pa}-p_{F_\bullet}(m) < \mathrm{pa}-p_{E_\bullet}(m), \quad (\textit{resp.}\ \mathrm{pa}-p_{F_\bullet}(m) \leqslant \mathrm{pa}-p_{E_\bullet}(m))$$

for $m >> 0$.

Remark 3.11. *Similarly, we can define the slope μ-stability by*

$$\mathrm{pa}-\mu(F_\bullet)(m) < \mathrm{pa}-\mu(E_\bullet)(m), \quad (\textit{resp.}\ \mathrm{pa}-\mu(F_\bullet)(m) \leqslant \mathrm{pa}-\mu(E_\bullet)(m)).$$

Recall that a flat family of parabolic sheaves $E_\bullet$ over a scheme T is a coherent $\mathcal{O}_{S_T}$-module $E_\bullet$ such that E is $f_T : S_T \to T$ torsion free and all E/E_i are flat (hence E_i are all flat) over T. A flat family of parabolic Higgs sheaves is a pair $(E_\bullet, \phi)$ of a flat flat family of parabolic sheaves $E_\bullet$ over a scheme T and an $\mathcal{O}_{S_T}$-homomorphism

$$\phi : E \otimes E \otimes_S K_S$$

such that $\phi(E_\alpha) \subseteq E_\alpha \otimes_S K_S$ for all α.

The moduli functor is defined by: Fix Hilbert polynomials $\mathbf{H} = (H, H_1, \cdots, H_d)$ and $\alpha = (\alpha_1, \cdots, \alpha_d), 0 \leqslant \alpha_1 < \cdots < \alpha_d < 1$, the moduli functor

$$\mathcal{N}^{\mathbf{H},\alpha}_{\mathrm{pa},(S,D)} : \mathrm{Sch}/\kappa \to \mathit{Sets}$$

by

$$T \mapsto \left\{ (E_\bullet, \phi) \;\middle|\; \begin{array}{l} (E_\bullet, \phi) \text{ is a flat family of parabolic Higgs sheaves} \\ \text{on } S_T/T \text{ with property (\#).} \end{array} \right\} \Big/ \cong .$$

where (#) is the following property for the family of parabolic Higgs sheaves $(E_\bullet, \phi)$ on S_T.

(#): For every geometric point $t \in T$, $(E_{\bullet,t}, \phi_t)$ is parabolic semistable with

$$\chi(E_t(m)) = H(m); \quad \chi((E_t/F_{i+1}(E)_t)(m)) = H_i(m),$$

and $(F_\bullet)_t$ is the filtration consisting of ϕ-invariant subsheaves

$$E_t = F_0(E)_t \supset F_1(E)_t \supset \cdots \supset F_d(E)_t = E_t(-D).$$

$\cong$ is an equivalence relation:

$$(E_\bullet, \phi) \cong (E'_\bullet, \phi')$$

if and only if

(1) $(E_\bullet, \phi) \cong (E'_\bullet, \phi') \otimes_T L$ for line bundle L on T;
(2) there exist filtrations consisting of ϕ-invariant subsheaves

$$E = E^0 \supset E^1 \supset \cdots \supset E^m = 0$$

$$E = E'^0 \supset E'^1 \supset \cdots \supset E'^m = 0$$

such that for any geometric point $t \in T$, their restrictions to S_t gives a Jordan-Hölder filtration of $((E_t)_\bullet, \phi)$ and $((E'_t)_\bullet, \phi')$ respectively, and $\mathrm{gr}(E_\bullet, \phi) = \oplus_{i=0}^m ((E^i/E^{i+1})_\bullet, \phi_i)$ is T-flat and that $\mathrm{gr}(E_\bullet, \phi) \cong \mathrm{gr}(E'_\bullet, \phi') \otimes_T L$, for some invertible sheaf L on T.

From [55, Theorem 2.9, Theorem 4.6], the functor $\mathcal{N}_{\mathrm{pa}} := \mathcal{N}^{\mathbf{H},\alpha}_{\mathrm{pa},(S,D)}$ is represented by a quasi-projective coarse moduli space $\mathcal{N}_{\mathrm{pa}} = \mathcal{N}^{\mathbf{H},\alpha}_{\mathrm{pa},(S,D)}$ over κ. The stable part $\mathcal{N}^s_{\mathrm{pa}}$ is given by an open subspace $\mathcal{N}^s_{\mathrm{pa}} \subset \mathcal{N}_{\mathrm{pa}}$.

3.2.2. *Moduli of Higgs sheaves on root stacks.* Let $\mathcal{S} := \sqrt[d]{(S,D)}$ be the d-th root stack. By choosing the generating sheaf Ξ for $\mathcal{S}$, and fixing a Hilbert polynomial H, the moduli space $\mathcal{N} := \mathcal{N}_H$ of semistable Higgs sheaves (E, ϕ) on $\mathcal{S}$ can be similarly defined. Here $E \in \mathrm{Coh}(\mathcal{S})$ is a torsion free sheaf, and

$$\phi : E \to E \otimes K_{\mathcal{S}}$$

is a section called the Higgs field. The Gieseker stability can be similarly defined by using the modified Hilbert polynomial $p_\Xi(E)$ associated with the generating sheaf Ξ, and ϕ-invariant subsheaves $F \subset E$.

In this section we show that the moduli space $\mathcal{N} := \mathcal{N}_H$ of semistable Higgs sheaves (E, ϕ) on $\mathcal{S}$ is isomorphic to the moduli space $\mathcal{N}_{\mathrm{pa}}$ of semistable parabolic Higgs sheaves $(E_\bullet, \phi)$ on (S, D), thus generalizing the result of [46], [6].

We first generalize the two functors $\mathcal{F}_{\mathcal{S}}, G_{\mathcal{S}}$ before to the abelian category of Higgs sheaves. Let $\mathrm{Higg}_{\mathcal{S}}$ be the abelian category of Higgs sheaves (E, ϕ) on $\mathcal{S}$, and let $\mathrm{Higg}^{\mathrm{pa}}_{(S,D)}$ be the abelian category of parabolic Higgs sheaves $(E_\bullet, \phi)$ on (S, D). There exists a functor

$$\mathcal{F}^{\phi}_{\mathcal{S}} : \mathrm{Higg}_{\mathcal{S}} \to \mathrm{Higg}^{\mathrm{pa}}_{(S,D)} \tag{3.2.1}$$

such that (E, ϕ) is mapped to an element in $\mathrm{Higg}^{\mathrm{pa}}_{(S,D)}$ as follows:

(1) $E \mapsto \mathcal{F}_{\mathcal{S}}(E) = p_*(E \otimes \mathcal{O}_{\mathcal{S}}(-l\mathcal{D}_{\mathrm{red}}))$ where $p : \mathcal{S} \to S$ is the map to its coarse moduli space, and $\mathcal{D} = p^{-1}(D)$;

(2) the section $\phi : E \to E \otimes K_{\mathcal{S}}$ will induce a section

$$\phi : \mathcal{F}_{\mathcal{S}}(E) \to \mathcal{F}_{\mathcal{S}}(E) \otimes K_{\mathcal{S}}$$

which can be obtained as follows.

We can take $\mathcal{D}_{\mathrm{red}}^k$ the k-th order infinitesimal neighborhood of $\mathcal{D}_{\mathrm{red}}$. Then $\mathcal{D}_{\mathrm{red}}^r = \mathcal{D}$. From [7, Equation 3.44],

$$K_{\mathcal{S}} = p^*(K_S \otimes \frac{d-1}{d}\mathcal{O}_S(D)).$$

Then the section $\phi : E \to E \otimes K_{\mathcal{S}}$ induces a morphism for any l

$$\phi : E \otimes \mathcal{O}_{\mathcal{S}}(-l\mathcal{D}_{\mathrm{red}}) \to E \otimes \mathcal{O}_{\mathcal{S}}(-l\mathcal{D}_{\mathrm{red}}) \otimes K_{\mathcal{S}}.$$

Then applying p_*:

$$\begin{aligned} p_*(E \otimes \mathcal{O}_{\mathcal{S}}(-l\mathcal{D}_{\mathrm{red}}) \otimes K_{\mathcal{S}}) &= p_*(E \otimes \mathcal{O}_{\mathcal{S}}(-l\mathcal{D}_{\mathrm{red}}) \otimes p^*(K_S \otimes \frac{d-1}{d}\mathcal{O}_S(D))) \\ &= p_*(E \otimes \mathcal{O}_{\mathcal{S}}((d-1) - l\mathcal{D}_{\mathrm{red}}) \otimes p^* K_S), \end{aligned}$$

and we get a morphism

$$p_*\phi : p_*(E \otimes \mathcal{O}_{\mathcal{S}}(-l\mathcal{D}_{\mathrm{red}})) \to p_*(E \otimes \mathcal{O}_{\mathcal{S}}(-l\mathcal{D}_{\mathrm{red}}) \otimes K_{\mathcal{S}}).$$

Since $r - 1 - l \geqslant -l$, this morphism must factor through

$$\begin{aligned} p_*(E \otimes \mathcal{O}_{\mathcal{S}}(-l\mathcal{D}_{\mathrm{red}})) &\to p_*(E \otimes \mathcal{O}_{\mathcal{S}}(-l\mathcal{D}_{\mathrm{red}}) \otimes p^* K_S) \\ &\to p_*(E \otimes \mathcal{O}_{\mathcal{S}}((d-1) - l\mathcal{D}_{\mathrm{red}}) \otimes p^* K_S), \end{aligned}$$

where $\mathcal{O}_{\mathcal{S}}(-l\mathcal{D}_{\mathrm{red}}) \to \mathcal{O}_{\mathcal{S}}(-l\mathcal{D}_{\mathrm{red}})$ is induced from the canonical section of divisors. Hence we get the section

$$\phi : p_*(E \otimes \mathcal{O}_{\mathcal{S}}(-l\mathcal{D}_{\mathrm{red}})) \to p_*(E \otimes \mathcal{O}_{\mathcal{S}}(-l\mathcal{D}_{\mathrm{red}}) \otimes K_S)$$

for any l, thus a section

$$\phi : \mathcal{F}_{\mathcal{S}}(E) \to \mathcal{F}_{\mathcal{S}}(E) \otimes K_S.$$

On the other hand, the functor

$$G_{\mathcal{S}}^{\phi} : \mathrm{Higg}_{(S,D)}^{\mathrm{pa}} \to \mathrm{Higg}_{\mathcal{S}}$$

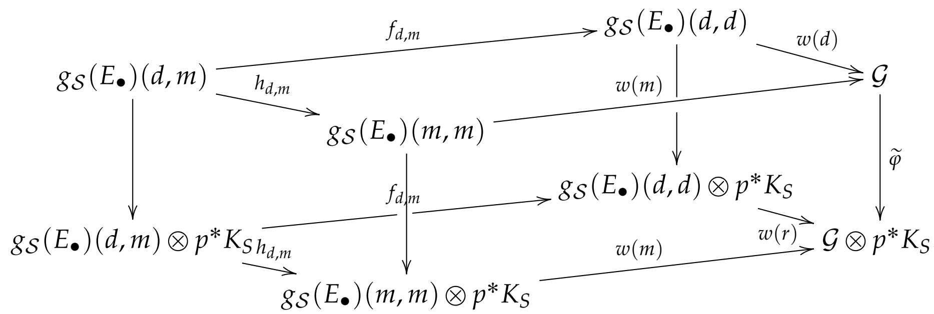
is given as follows. First we have for a pair $(E_\bullet, \phi) \in \mathrm{Higg}_{(S,D)}^{\mathrm{pa}}$,

$$E_\bullet \mapsto \int^{\mathbb{Z}} g_{\mathcal{S}}(E_\bullet(d,d)).$$

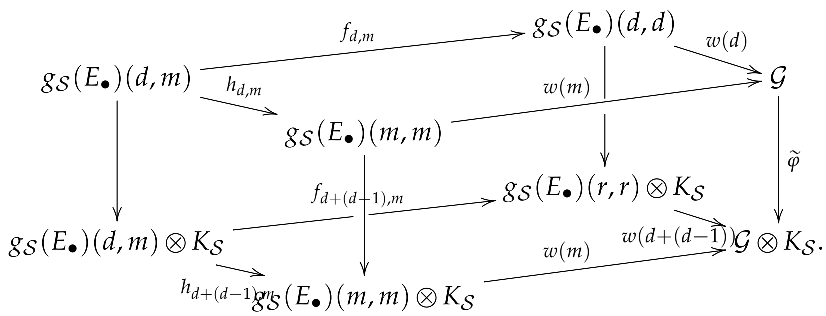
We construct a Higgs field on $\int^{\mathbb{Z}} g_{\mathcal{S}}(E_\bullet(d,d))$. The Higgs field on $E_\bullet$ is $\phi : E_\bullet \to E_\bullet \otimes K_S$. Therefore

$$p^*\phi : (p^*E_\bullet) \otimes \mathcal{O}_{\mathcal{S}}(l\mathcal{D}_{\mathrm{red}}) \to (p^*E_\bullet) \otimes \mathcal{O}_{\mathcal{S}}(l\mathcal{D}_{\mathrm{red}}) \otimes p^* K_S$$

gives the Higgs field on the pullback. From the diagram of wedges:
(3.2.2)



Since $K_{\mathcal{S}} = p^*K_S \otimes p^*(\frac{d-1}{d}\mathcal{O}_S(D)) = p^*K_S \otimes \mathcal{O}_{\mathcal{S}}((d-1)\mathcal{D}_{\text{red}})$, we have the following diagram:
(3.2.3)



and the vertical arrows must factor through the vertical arrows in (3.2.2). Therefore by taking the colimit, we have a Higgs field on the coend:

$$\phi : \int^{\mathbb{Z}} g_{\mathcal{S}}(E_\bullet(d,d)) \to \int^{\mathbb{Z}} g_{\mathcal{S}}(E_\bullet(d,d)) \otimes K_{\mathcal{S}}.$$

Recall that a parabolic sheaf $E_\bullet \in \mathrm{Par}_{\frac{1}{d}}(S,D)$ is torsion free if $E = E_0$ is torsion free. Thus the functor $G_{\mathcal{S}}^{\phi}$ sends torsion free sheaves on S to torsion free sheaves on the corresponding root stack $\mathcal{S}$.

Proposition 3.12. *The functor $\mathcal{F}_{\mathcal{S}}^{\phi}$ sends flat families of torsion free Higgs sheaves on $\mathcal{S}$ to flat families of torsion free parabolic Higgs sheaves on S. Similarly, the functor $G_{\mathcal{S}}^{\phi}$ sends flat families of torsion free parabolic Higgs sheaves on S to flat families of torsion free Higgs sheaves on the root stack $\mathcal{S}$.*

Proof. This is result is proved for the functors $\mathcal{F}_{\mathcal{S}}$ and $G_{\mathcal{S}}$ from [46, Lemma 7.9]. It is enough to check the Higgs fields $\phi : E \to E \otimes K_{\mathcal{S}}$ and $\phi : E_\bullet \to E_\bullet \otimes K_S$ for Higgs pairs. Since the Higgs fields in the functors $\mathcal{F}_{\mathcal{S}}^{\phi}$ and $G_{\mathcal{S}}^{\phi}$ are preserved by families and we are done. □

Therefore for the generating sheaf $\Xi = \oplus_{l=0}^{d-1}\mathcal{O}_{\mathcal{S}}(l\mathcal{D}_{\text{red}})$, the Hilbert polynomials (3.1.2) of the Higgs pairs on the root stack $\mathcal{S}$ is the same as the parabolic Hilbert polynomial (3.1.1). Thus the Gieseker stability of Higgs pairs and parabolic Higgs pairs are equivalent by choosing ϕ-invariant sub Higgs sheaves. Therefore we have the following result on the moduli spaces:

Theorem 3.13. *Let $\mathcal{S} = \sqrt[d]{(S, D)}$ be the root stack of S with respect to the smooth divisor D. Choosing the generating sheaf $\Xi = \oplus_{l=0}^{d-1}\mathcal{O}_{\mathcal{S}}(l\mathcal{D}_{\mathrm{red}})$, and fixing some modified Hilbert polynomial $H \in \mathbb{Q}[m]$. Then the moduli space $\mathcal{N}_H := \mathcal{N}_H(\mathcal{S})$ of semistable Higgs pairs on the root stack $\mathcal{S}$ is isomorphic to the moduli space $\mathcal{N}_{\mathrm{pa}} := \mathcal{N}^{H,\alpha}_{\mathrm{pa},(S,D)}$ of semistable parabolic Higgs pairs on (S, D). Their corresponding stable open subspaces $\mathcal{N}^s_H$ and $\mathcal{N}^s_{\mathrm{pa}}$ are also isomorphic.* □

4. Moduli stack of relative Higgs pairs

In this section we define the relative moduli space of Higgs pairs on the root stack $\mathcal{S}$ with respect to the stacky divisor $\mathcal{D}$. We relate it to the relative parabolic Higgs pairs on (S, D), relative moduli space of parabolic stable sheaves of Kapranov and the Vafa-Witten invariants for the moduli space of Higgs sheaves on $\mathcal{S}$.

4.1. **Relative moduli space, after Kapranov.** Let $\mathcal{S} = \sqrt[d]{(S, D)}$ be a d-th root stack associated with the smooth divisor $D \subset S$, and $\mathcal{D} = p^{-1}(D)$ be the stacky divisor. Let $\widehat{\mathcal{S}}_{\mathcal{D}}$ be the formal completion of $\mathcal{S}$ along the divisor $\mathcal{D} \subset \mathcal{S}$, i.e., if $\mathcal{I}_{\mathcal{D}}$ is the ideal sheaf of $\mathcal{D}$ in $\mathcal{S}$, then

$$\widehat{\mathcal{S}}_{\mathcal{D}} = \varinjlim_n (\mathcal{S}/\mathcal{I}^n_{\mathcal{D}}).$$

Then $\widehat{\mathcal{S}}_{\mathcal{D}}$ is a stftformal scheme over the discrete valuation ring $\mathrm{Spf}(R)$. We can understand this formal scheme $\widehat{\mathcal{S}}_{\mathcal{D}}$ as the formal thickening of $\mathcal{D} \subset \mathcal{S}$ and also the formal neighborhood of $\mathcal{D}$ in $\mathcal{S}$, as in [34, §5]. The central fibre if a $\kappa = R/(t)$ stack $\mathcal{D}$, and the generic fibre $\widehat{\mathcal{S}}_{\mathcal{D}} \backslash \mathcal{D} = (\widehat{\mathcal{S}}_{\mathcal{D}})_\eta$ is a non-archimedean space over the non-archimedean field $\mathbb{K} = \kappa((t))$.

4.1.1. *Formal moduli of relative sheaves.* For the root stack $\mathcal{S}$, the generating sheaf $\Xi = \oplus_{l=0}^{r-1}\mathcal{O}_{\mathcal{S}}(l\mathcal{D}_{\mathrm{red}})$. Since $\widehat{\mathcal{S}}_{\mathcal{D}}$ is the formal completion of $\mathcal{S}$ along $\mathcal{D}$, from the relative GAGA [9], the generating sheaf Ξ gives a generating sheaf $\widehat{\Xi}$ on $\widehat{\mathcal{S}}_{\mathcal{D}}$. From [28, §4], when fixing some Hilbert polynomial H associated with the generating sheaf, we have the formal moduli stack $\widehat{\mathscr{M}_R}(\widehat{\mathcal{S}}_{\mathcal{D}}, H)$ of stable coherent sheaves on $\widehat{\mathcal{S}}_{\mathcal{D}}$ with Hilbert polynomial H.

We define a relative version of the moduli space of stable sheaves. Fix a stable sheaf $\widehat{E}^0$ on $\widehat{\mathcal{S}}_{\mathcal{D}} \backslash \mathcal{D}$, let $\widehat{\mathscr{M}^H_R}(\widehat{\mathcal{S}}_{\mathcal{D}}; \mathcal{D})$ be the moduli functor that sends a formal scheme $\mathfrak{T}$ over $\mathrm{Spf}(R)$ to the isomorphic classes of families of stable sheaves $\widehat{E}$ on $\widehat{\mathcal{S}}_{\mathcal{D}}$ with Hilbert polynomial H such that

$$\widehat{E}|_{(\widehat{\mathcal{S}}_{\mathcal{D}})_\eta} \cong \widehat{E}^0.$$

Theorem 4.1. *The moduli stack $\widehat{\mathscr{M}^H_R}(\widehat{\mathcal{S}}_{\mathcal{D}}; \mathcal{D})$ is a subformal moduli stack of $\widehat{\mathscr{M}_R}(\widehat{\mathcal{S}}_{\mathcal{D}}, H)$.*

Proof. From [24, Theorem 4.10], we have

$$(\widehat{\mathscr{M}_R}(\widehat{\mathcal{S}}_{\mathcal{D}}, H))_\eta \overset{\psi}{\cong} \mathscr{M}_{\mathbb{K}}((\widehat{\mathcal{S}}_{\mathcal{D}})_\eta, H),$$

where $\mathscr{M}_{\mathbb{K}}((\hat{\mathcal{S}}_{\mathcal{D}})_\eta, H)$ is the moduli space of stable coherent sheaves on the $\mathbb{K}$-analytic space $(\hat{\mathcal{S}}_{\mathcal{D}})_\eta$ with Hilbert polynomial H. Since the fixed point $\widehat{E}^0 \in \mathscr{M}_{\mathbb{K}}((\hat{\mathcal{S}}_{\mathcal{D}})_\eta, H)$, then

$$\psi^{-1}(\widehat{E}^0) \cong (\widehat{\mathscr{M}_R}(\hat{\mathcal{S}}_{\mathcal{D}}, H))_\eta$$

and $\widehat{\mathscr{M}_R^H}(\hat{\mathcal{S}}_{\mathcal{D}}; \mathcal{D})$ is a formal substack of $\widehat{\mathscr{M}_R}(\hat{\mathcal{S}}_{\mathcal{D}}, H)$. □

Following [34, §(2.2)], for such a pair $(\mathcal{S}, \mathcal{D})$, and a fixed E^0 on $\mathcal{S}^0 := \mathcal{S}\backslash\mathcal{D}$, we have the following moduli functor:

$$\mathscr{M}_R(\mathcal{S}; \mathcal{D}, H) : \mathrm{Sch}_\kappa \to \left\{ E \,\middle|\, \begin{array}{l} E \text{ is a flat family of coherent sheaves} \\ \text{on } S_T \text{ such that } E|_{(\mathcal{S}\backslash\mathcal{D})_T \cong (E^0)_{\mathcal{S}^0_T}}. \end{array} \right\} \Big/ \cong .$$

Kapranov [34, §(2.2)] proves that the functor $\mathscr{M}_R(\mathcal{S}; \mathcal{D}, H)$ is represented by an ind-scheme which we still denote it by $\mathscr{M}_R(\mathcal{S}; \mathcal{D}, H)$.

Lemma 4.2. *The ind-scheme $\mathscr{M}_R(\mathcal{S}; \mathcal{D}, H)$ can be extended to a formal scheme $\mathscr{M}_R\widehat{(\mathcal{S}; \mathcal{D}}, H)$ over Spf(R).*

Proof. We can assume that the fixed E^0 on $\mathcal{S}^0 := \mathcal{S}\backslash\mathcal{D}$ has fixed determinant $\det(E^0) = \mathcal{O}_{\mathcal{S}^0}$. Then if E is such a stable coherent sheaf on $\mathcal{S}$ such that $E|_{\mathcal{S}^0} = E^0$, let

$$\iota : \mathcal{S}^0 \hookrightarrow \mathcal{S}$$

be the inclusion and $\mathfrak{m}$ the ideal sheaf of $\mathcal{D}$ in $\mathcal{O}_{\mathcal{S}}$, then

$$E \subset \iota_* E^0$$

must be contained in $\mathfrak{m}^{-N}E^0/\mathfrak{m}^N E^0$ for some $N >> 0$, therefore a closed subscheme, which we denote it by

$$\mathscr{M}_R(\mathcal{S}; \mathcal{D}, H)_N,$$

who contains all the stable sheaves E on $\mathcal{S}$ such that $E \subset \iota_* E^0$ is in the Quot scheme $\mathrm{Quot}(\mathfrak{m}^{-N}E^0/\mathfrak{m}^N E^0)$. If we consider all such $N \in \mathbb{Z}$, then we get a direct limit

$$\varinjlim_N \mathscr{M}_R(\mathcal{S}; \mathcal{D}, H)_N$$

of all such closed subschemes. Let us take a base change to Spf(R) from Spf(κ), then we get a formal scheme $\mathscr{M}_R\widehat{(\mathcal{S}; \mathcal{D}}, H)$ over Spf(R). □

4.1.2. *Relative parabolic moduli space.* In this section we define the parabolic version of the relative moduli space of stable sheaves. Fix a pair (S, D), $S^0 := S\backslash D$ and a stable coherent sheaf E^0 on S^0 again, let $\mathscr{M}_{\mathrm{pa}}^{H,\alpha}(S; D)$ be the moduli stack functor of parabolic relative stable sheaves with Hilbert polynomial H, i.e., it is the functor from the category of schemes to groupoids that sends a scheme T over κ to the families of the following triples (E, τ, π) over T:

1. (1) E is a family of stable shaves over S with Hilbert polynomial H;
2. (2) $\tau : E|_{S^0} \cong E^0$ is an isomorphism;
3. (3) π is a parabolic structure on $E|_D$ as in Definition 3.1.

From [34, Corollary (3.1.2)], $\mathscr{M}_{\mathrm{pa}}^{H,\alpha}(S; D)$ is represented by a fine moduli space if the intersection number D_S^2 is negative. In general, similar to Lemma 4.2, $\mathscr{M}_{\mathrm{pa}}^{H,\alpha}(S; D)$ can be extended to a formal scheme over Spf(R).

Remark 4.3. *The method to prove the above result is as follows. First there exists a formal moduli scheme* $\widehat{\mathscr{M}}^{H,\alpha}_{R,\mathrm{pa}}(\widehat{S}_D;D)$ *of stable sheaves* $\widehat{E}$ *on* $\widehat{S}_X$ *such that* $\widehat{E}|_{S^0}\cong E^0$ *and* π *is a parabolic structure on* $\widehat{E}|_D$. *Similar proof as in Lemma 4.2 gives the result.*

4.2. **Formal and relative moduli of Higgs pairs.** Let $\widehat{S}$ be a stft formal scheme over R and let us fix a Hilbert polynomial $H\in\mathbb{Q}[m]$. In [28, §4] the formal moduli scheme $\widehat{\mathscr{M}}^{H}_{R}(\widehat{S})$ of stable sheaves $\widehat{E}$ with Hilbert polynomial H is defined and constructed. We generalize this construction to the formal moduli of Higgs pairs.

Let us first define the Higgs pairs over a locally Noetherian scheme. Let $\mathcal{T}$ be a locally noetherian scheme, and $\mathrm{Sch}_{\mathcal{T}}$ the category of coherent sheaves over $\mathcal{T}$.

Definition 4.4. *Let* $\mathfrak{X}/\mathcal{T}$ *be a projective scheme and* $\mathcal{O}_{\mathfrak{X}}(1)$ *the Serre line bundle. Fix a Hilbert polynomial* $H\in\mathbb{Q}[m]$, *define the functor*

$$\mathcal{N}^{H}_{\mathcal{T},\mathrm{pa}}(\mathfrak{X}):\mathrm{Sch}_{\mathcal{T}}\to \mathit{Sets}$$

such that

$$\{\mathcal{T}'\to\mathcal{T}\}\mapsto\left\{E_\bullet\;\middle|\;\begin{array}{l}\bullet: E_\bullet \text{ is a flat family of coherent sheaves}\\ \quad\text{on } \mathcal{T}' \text{ with Hilbert polynomial } H;\\ \bullet:\phi: E_\bullet\to E_\bullet\otimes K_{\mathcal{T}'} \text{ a Higgs field.}\end{array}\right\}\Big/\cong.$$

Here the equivalence relation $\cong$ *means that*

$$(E_\bullet,\phi)\sim(E'_\bullet,\phi')$$

if and only if

$$(E_\bullet,\phi)\cong(E'_\bullet,\phi')\otimes_{\mathcal{T}'}\mathcal{L}$$

for some $\mathcal{L}\in\mathrm{Pic}(\mathfrak{X})$.

Theorem 4.5. *The functor* $\mathcal{N}^{H}_{\mathcal{T},\mathrm{pa}}(\mathfrak{X})$ *is an algebraic stack locally of finite type over* $\mathcal{T}$.

Proof. We need to check the conditions (1), (2), (3), (4) of Theorem 5.3 in [1], and the checking is the same as [28, Theorem 3.3]. We omit the details. □

Let $\mathcal{T}_m:=\mathrm{Spec}(R/(t^{m+1}))$, where t is a uniformizer of the discrete valuation ring R. Then from [28, Proposition 4.8], we have

Theorem 4.6. *Let* $\widehat{S}$ *be a stft formal scheme over* R, *and let*

$$\widehat{S}_m:=\widehat{S}\times_R\mathcal{T}_m.$$

Then

$$\varinjlim_m \mathcal{N}^{H}_{\mathcal{T}_m,\mathrm{pa}}(\widehat{S}_m)\cong\mathcal{N}^{H}_{R,\mathrm{pa}}(\widehat{S}),$$

where $\mathcal{N}^{H}_{R,\mathrm{pa}}(\widehat{S})$ *is the formal moduli stack of stable parabolic formal sheaves over* $\widehat{S}$ *with Hilbert polynomial* H.

Corollary 4.7. *(*[28, Theorem 4.10]*) We have*

$$(\mathcal{N}^{H}_{R,\mathrm{pa}}(\widehat{S}))_\eta\cong\mathcal{N}^{H}_{\mathbb{K},\mathrm{pa}}(\widehat{S}_\eta)$$

where $\mathcal{N}^{H}_{\mathbb{K},\mathrm{pa}}(\widehat{S}_\eta)$ *is the moduli stack of stable parabolic coherent sheaves over the* $\mathbb{K}$-*analytic space* $\widehat{S}_\eta$ *with Hilbert polynomial* H.

The formal relative moduli space of parabolic Higgs pairs can be similarly defined. Let $\widehat{S}_D$ be the formal completion of S along the divisor D, and let

$$\widehat{S}_D = \widehat{S}_D^0.$$

The formal relative moduli stack $\mathcal{N}^H_{R,\mathrm{pa}}(\widehat{S};D)$ is the formal moduli scheme of formal parabolic stable sheaves $\widehat{E}_\bullet$ on $\widehat{S}_D$ such that

$$\widehat{E}_\bullet|_{\widehat{S}_D^0} \cong \widehat{E}^0$$

for a fixed $\widehat{E}^0$ over $\widehat{S}_D^0$. The same arguments as in Theorem 4.1 shows that $\mathcal{N}^H_{R,\mathrm{pa}}(\widehat{S};D)$ is a formal subscheme of $\mathcal{N}^H_{R,\mathrm{pa}}(\widehat{S})$.

5. Relation to the Vafa-Witten invariants

In this section we relate the relative moduli spaces of stable sheaves and relative moduli space of Higgs sheaves on a pair (S,D) to the Vafa-Witten invariants for (S,D) and the Eisenstein series of the associated curve D.

In Section 5.1 and Section 5.2 we borrow the notation $X = D$ to represent a smooth projective curve.

5.1. **Eisenstein series.** We review the basic materials of the classical unramified geometric theory of Eisenstein series following [20], [37], which was recalled in [34, §4].

Let $r \geqslant 1$ be an integer, and X a smooth projective curve over a finite field $\mathbb{F}_q$. We denote by $F = \mathbb{F}_q(X)$ the function field of X. Let $\mathrm{Bun}^r_{X/\mathbb{F}_q}$ be the moduli stack of rank r vector bundles $\mathcal{L}$ on X, which is an algebraic stack locally of finite type. The connected components of $\mathrm{Bun}^r_{X/\mathbb{F}_q}$ is given by a stack

$$\mathrm{Bun}^{r,l}_{X/\mathbb{F}_q}$$

of locally of finite type and degree l. The stack $\mathrm{Bun}^{r,l}_{X/\mathbb{F}_q}$ is smooth of dimension $r^2(g-1)$, where $g = g_X$ is the genus of X.

We consider the stack over $\mathbb{F}_q$:

$$\mathrm{Bun}_{X/\mathbb{F}_q,(1^r)}$$

which classifies the flags

$$\mathcal{L}_\bullet = (0 = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_r = \mathcal{L})$$

where each $\mathcal{L}_i$ is a vector bundle of rank i on X, locally direct factor of $\mathcal{L}_{i+1}$ as an $\mathcal{O}_X$-locally free module for $i = 0,\cdots,r-1$. This stack is also an algebraic stack over $\mathbb{F}_q$, with connected components parametrized by:

$$\underline{\lambda} = (\lambda_1,\cdots,\lambda_r) \in \mathbb{Z}^r$$

where

$$\mathrm{Bun}^{\underline{\lambda}}_{X/\mathbb{F}_q,(1^r)}$$

classifies the $\mathcal{L}_\bullet$ flags as above such that $\mathcal{A}_i = \mathcal{L}_i/\mathcal{L}_{i+1}$ is an invertible $\mathcal{O}_X$-module of degree λ_i for $i = 1,\cdots,r$. The stack $\mathrm{Bun}_{X/\mathbb{F}_q,(1^r)}$ is also smooth, locally of finite type over $\mathbb{F}_q$ and

$$\dim(\mathrm{Bun}^{\underline{\lambda}}_{X/\mathbb{F}_q,(1^r)}) = \frac{r(r+1)}{2}(g-1) + \sum_{1\leqslant i<j\leqslant r}(\lambda_j - \lambda_i).$$

We have the following morphisms of stacks:

$$\begin{array}{ccc} \mathrm{Bun}_{X/\mathbb{F}_q,(1^r)} & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\rho} & \\ \mathrm{Bun}^r_{X/\mathbb{F}_q} & & (\mathrm{Pic}_{X/\mathbb{F}_q})^r \end{array}$$

by:

$$\pi(\mathcal{L}_\bullet) = \mathcal{L};$$

$$\rho(\mathcal{L}_\bullet) = (\mathcal{L}_1/\mathcal{L}_0, \cdots, \mathcal{L}_r/\mathcal{L}_{r-1}) = (\mathcal{A}_1, \cdots, \mathcal{A}_r).$$

We denote by

$$\pi^{\underline{\lambda}} : \mathrm{Bun}^{\underline{\lambda}}_{X/\mathbb{F}_q,(1^r)} \to \mathrm{Bun}^{r,l}_{X/\mathbb{F}_q}$$

and

$$\rho^{\underline{\lambda}} : \mathrm{Bun}^{\underline{\lambda}}_{X/\mathbb{F}_q,(1^r)} \to \prod_{i=1}^{r} \mathrm{Pic}^{\lambda_i}_{X/\mathbb{F}_q}$$

the restrictions of π and ρ respectively to $\mathrm{Bun}^{\underline{\lambda}}_{X/\mathbb{F}_q,(1^r)}$ where $l = \lambda_1 + \cdots + \lambda_r$. The morphism π is representable and locally of finite type and in fact $\pi^{\underline{\lambda}}$ is representable and quasi-projective for each $\underline{\lambda} \in \mathbb{Z}^r$.

Recall the ring of adeles $\mathbb{A}_F$ of F, and the ring of integers O_F, then Weil's observation shows that

$$\mathrm{Bun}^r_{X/\mathbb{F}_q}(\mathbb{F}_q) \xrightarrow{\sim} \left[GL_r(F)\backslash GL_r(\mathbb{A}_F)/GL_r(O_F)\right]$$

and

$$\mathrm{Bun}_{X/\mathbb{F}_q,(1^r)}(\mathbb{F}_q) \xrightarrow{\sim} \left[B_r(F)\backslash GL_r(\mathbb{A}_F)/GL_r(O_F)\right]$$

where $B_r(F) \subset GL_r(F)$ is the Borel subgroup of upper triangular matrices.

Moreover, π is identified with the canonical projection:

$$\left[B_r(F)\backslash GL_r(\mathbb{A}_F)/GL_r(O_F)\right] \to \left[GL_r(F)\backslash GL_r(\mathbb{A}_F)/GL_r(O_F)\right]$$

Finally, the inclusion $B_r \subset GL_r$ induces an equivalence of categories

$$\left[B_r(F)\backslash B_r(\mathbb{A}_F)/GL_r(O_F)\right] \to \left[GL_r(F)\backslash GL_r(\mathbb{A}_F)/GL_r(O_F)\right]$$

and ρ is identified with the projection

$$\left[B_r(F)\backslash B_r(\mathbb{A}_F)/GL_r(O_F)\right] \to \left[F^\times\backslash \mathbb{A}_F^\times/O_F^\times\right]^r$$

by

$$B_r(F)\cdot b\cdot B_r(O_F) \mapsto (F^\times b_{11}O_F^\times, \cdots, F^\times b_{nn}O_F^\times).$$

We define Eisenstein series. For each $\mathcal{L} \in \mathrm{Bun}^r_{X/\mathbb{F}_q}(\mathbb{F}_q)$, let $\Gamma(\mathcal{L};\underline{\lambda})$ be the scheme of B-structures on $\mathcal{L}$. This is the fibre of $\pi^{\underline{\lambda}}$ above $\mathcal{L} \in \mathrm{Bun}^{r,l}_{X/\mathbb{F}_q}$. The Eisenstein series is the generating function

$$E_{\mathcal{L}}(z) = \sum_{\substack{\underline{\lambda}\in\mathbb{Z}^r,\\ l=\lambda_1+\cdots+\lambda_r}} \chi(\Gamma(\mathcal{L};\underline{\lambda}))z^l \tag{5.1.1}$$

We also introduce another formula as in [37, §3]. Let

$$\overline{\mathrm{Bun}}_{X/\mathbb{F}_q,(1^r)}$$

be the stack of generalized flags

$$\mathcal{L}'_\bullet = (0 = \mathcal{L}'_0 \subset \mathcal{L}'_1 \subset \cdots \subset \mathcal{L}'_r = \mathcal{L})$$

where each $\mathcal{L}'_j$ is a locally free $\mathcal{O}_X$-module rank j on X, but not necessarily locally a direct factor of $\mathcal{L}'_{j+1}$ for $ij = 0, \cdots, r-1$. This is an algebraic stack locally of finite type and smooth over $\mathbb{F}_q$, containing $\mathrm{Bun}_{X/\mathbb{F}_q,(1^r)}$ as an open dense subset. Let

$$\begin{array}{ccc} \overline{\mathrm{Bun}}_{X/\mathbb{F}_q,(1^r)} & & \\ \downarrow{\scriptstyle \overline{\pi}} & \searrow{\scriptstyle \overline{\rho}} & \\ \mathrm{Bun}^r_{X/\mathbb{F}_q} & & (\mathrm{Pic}_{X/\mathbb{F}_q})^r \end{array}$$

be the two morphisms of stacks over $\mathbb{F}_q$ defined as follows:

$$\overline{\pi}(\mathcal{L}'_\bullet) = \mathcal{L}'_r = \mathcal{L};$$

$$\overline{\rho}(\mathcal{L}'_\bullet) = (\det(\mathcal{A}'_1), \cdots, \det(\mathcal{A}'_r))$$

where for $i = 1, \cdots, r$, $\mathcal{A}'_j = \mathcal{L}'_j/\mathcal{L}'_{j-1}$ is a coherent $\mathcal{O}_X$-module of generic rank 1 and

$$\det(\mathcal{A}'_j) = (\Lambda^j\mathcal{L}'_j) \otimes (\Lambda^{j-1}\mathcal{L}'_{j-1})^{\otimes -1}$$

is the invertible determinant $\mathcal{O}_X$-module of $\mathcal{A}'_j$.

The connected components of $\overline{\mathrm{Bun}}_{X/\mathbb{F}_q,(1^r)}$ are still parametrized by $\mathbb{Z}^r$ and we denote by

$$\overline{\pi}^{\underline{\lambda}'} : \overline{\mathrm{Bun}}^{\underline{\lambda}'}_{X/\mathbb{F}_q,(1^r)} \to \mathrm{Bun}^{r,\lambda'_1+\cdots+\lambda'_r}_{X/\mathbb{F}_q}$$

and

$$\overline{\rho}^{\underline{\lambda}'} : \overline{\mathrm{Bun}}^{\underline{\lambda}'}_{X/\mathbb{F}_q,(1^r)} \to \prod_{i=1}^r \mathrm{Pic}^{\lambda'_i}_{X/\mathbb{F}_q}$$

the restrictions of $\overline{\pi}$ and $\overline{\rho}$ respectively, with the connected components indexed by $\underline{\lambda}' \in \mathbb{Z}^r$.

The morphism $\overline{\pi}$ is representable and locally of finite type and in fact, $\overline{\pi}^{\underline{\lambda}'}$ is representable and projective for each $\underline{\lambda}' \in \mathbb{Z}^r$.

We give the definition of modified Eisenstein series. Fix a $\mathcal{L} \in \mathrm{Bun}^{r,l}_{X/\mathbb{F}_q}$, let $\overline{\Gamma}(\mathcal{L}, \underline{\lambda}')$ be the scheme of generalized flags over $\mathcal{L}$, which is the fibre of $\overline{\pi}^{\underline{\lambda}'}$. It is a scheme over $\mathbb{F}_q$. Define the "modified Eisenstein series" by:

$$\overline{E}_{\mathcal{L}}(z) = \sum_{\substack{\underline{\lambda}' \in \mathbb{Z}^r, \\ l = \lambda_1' + \cdots + \lambda_r'}} \chi(\overline{\Gamma}(\mathcal{L}; \underline{\lambda}')) z^l \tag{5.1.2}$$

Here are two results in the rank two case.

Proposition 5.1. [37] *We have:*

$$\overline{E}_{\mathcal{L}}(z_1, z_2) = \sum_{(a_1, a_2) \in \mathbb{Z}^2} \chi(\overline{\Gamma}_{a_1, a_2}(\mathcal{L}) z_1^{a_1} z_2^{a_2} = \zeta(z_2/z_1) \cdot E_{\mathcal{L}}(z_1, z_2),$$

where $\zeta(q)$ is the zeta function.

Proposition 5.2. [34, Proposition 4.2.2.] *The series $\overline{E}_{\mathcal{L}}(z_1, z_2)$ represents a rational function with only poles being simple poles along the line $z_1 = z_2$. The series satisfies the functional equation:*

$$\overline{E}_{\mathcal{L}}(z_1, z_2) = (z_1/z_2)^{2-2g} \overline{E}_{\mathcal{L}}(z_2, z_1).$$

Remark 5.3. *One can define the motivic version of Eisenstein series as in* [34], *and Proposition 5.2 can be proved for motivic Eisenstein series. We only deal with Euler characteristic in this paper, and leave the motivic Eisenstein series and how it is related to motivic Vafa-Witten invariants for future work.*

5.2. **Moduli of Higgs pair on curves.** In this section we review the moduli stack of Higgs pairs on the smooth curve X, which is the moduli space of Hitchin systems.

Let us fix a divisor $K = \sum_i n_i \cdot [x_i]$ on X which is effective. The Hitchin moduli stack $\mathcal{N}_K^r$ classifies pairs

$$(E, \phi)$$

where E is a rank r vector bundle on X, and

$$\phi \in \mathrm{Hom}(E, E \otimes \mathcal{O}_X(K))$$

is a section which is allowed to have a pole of order at most n_i at x_i. This stack $\mathcal{N}_K^r$ is an algebraic stack locally of finite type. If we fix some stability condition on (E, ϕ), the moduli stack will be a scheme.

Remark 5.4.
(1) *In some references* [18], [17], *the moduli stack of Hitchin system only consider the pairs (E, ϕ) such that $\phi \in Hom(E, E \otimes \omega_X)$. It is natural to study the generalized Higgs pairs on X, since the restriction of a Higgs pair (E, ϕ) on a projective surface (S, X) to X gives a generalized Higgs pair.*
(2) *There is a morphism from $\mathcal{N}_K^r$ to an affine space*

$$\mathcal{A}_K = \bigoplus_i H^0(X, \mathcal{O}_X((m_i + 1)K))$$

where the m_i's are the exponents of GL_r. This morphism is called the Hitchin fibration. We hope to return to this morphism later to see if it gives some new phenomenon of the Vafa-Witten invariants.

We let $\overline{\mathcal{N}}^K_{X/\mathbb{F}_q,(1^r)}$ be the stack of generalized flags

$$E'_\bullet = (0 = E'_0 \subset E'_1 \subset \cdots \subset E'_r = E)$$

such that $E'_\bullet \in \overline{\mathrm{Bun}}_{X/\mathbb{F}_q,(1^r)}$, (E,ϕ) is a Higgs pair in $\mathcal{N}^K_{X/\mathbb{F}_q}$, and the Higgs field ϕ gives a section $\phi : E'_i \to E'_i \otimes \mathcal{O}_X(K)$. Then there exist morphisms of stacks:

$$\begin{array}{ccc} \overline{\mathcal{N}}^K_{X/\mathbb{F}_q} & & \\ \big\downarrow \overline{\pi} & \overset{\overline{\rho}}{\searrow} & \\ \mathrm{Bun}^r_{X/\mathbb{F}_q} & & (\mathrm{Pic}_{X/\mathbb{F}_q})^r \end{array}$$

such that

$$\overline{\pi}(E'_\bullet,\phi) = E'_r = E$$
$$\overline{\rho}(E'_\bullet,\phi) = (\det(\mathcal{A}'_1),\cdots,\det(\mathcal{A}'_r))$$

as before. The connected components of $\overline{\mathcal{N}}^K_{X/\mathbb{F}_q}$ are still parametrized by $\mathbb{Z}^r$, and we denote by

$$\overline{\pi}^{\underline{\lambda}'} : \overline{\mathcal{N}}^{K,\underline{\lambda}'}_{X/\mathbb{F}_q,(1^r)} \to \mathrm{Bun}^{\lambda'_1+\cdots+\lambda'_r}_{X/\mathbb{F}_q}$$

and

$$\overline{\rho}^{\underline{\lambda}'} : \overline{\mathcal{N}}^{K,\underline{\lambda}'}_{X/\mathbb{F}_q,(1^r)} \to \prod_{j=1}^r \mathrm{Pic}^{\lambda'_j}_{X/\mathbb{F}_q}$$

the restriction of $\overline{\pi}$ and $\overline{\rho}$ respectively, with the connected components indexed by $\underline{\lambda}' \in \mathbb{Z}^r$.

The morphism $\overline{\pi}$ is representable and locally of finite type and in fact, $\overline{\pi}^{\underline{\lambda}'}$ is representable and quasi-projective for each $\underline{\lambda}' \in \mathbb{Z}^r$. Fix a $E \in \mathrm{Bun}^{r,l}_{X/\mathbb{F}_q}$, we let

$$\overline{\mathrm{N}}(E,\underline{\lambda}')$$

be the scheme of generalized flags over E with Higgs field ϕ, which is the fibre of $\overline{\pi}^{\underline{\lambda}'}$. We form the series

$$\overline{\mathrm{N}}_E(z) = \sum_{\substack{\underline{\lambda}' \in \mathbb{Z}^r,\\ l=\lambda'_1+\cdots+\lambda'_r}} \chi(\overline{\mathrm{N}}(E;\underline{\lambda}'))z^l \tag{5.2.1}$$

We call it the corresponding Eisenstein series of Higgs pairs. We have a generalization of functional equation for Eisenstein series.

Proposition 5.5. *In the case of rank $r = 2$, the series $\overline{\mathrm{N}}_E(z)$ represents a rational function with only poles being simple poles along $z_1 = z_2$. It satisfies the functional equation:*

$$\overline{\mathrm{N}}_E(z_1,z_2) = (z_1/z_2)^{2-2g}\overline{\mathrm{N}}_E(z_2,z_1).$$

Proof. The proof in [34, §(4.2.2)] works in the case of Higgs pairs. We provide an argument here in the motivic level. The measure μ is the motivic measure in the Grothendieck ring $K_0(\mathrm{Var}_\kappa)$ of κ-varieties, which is generated by $[Z]$ where Z is a quasi-projective variety. The relations are given by $[Z] = [Y] + Z - Y$ for a closed $Y \subset Z$ and $[Z_1]\cdot[Z_2] = [Z_1 \times Z_2]$. We use $\mathbb{L}$ to represent the motivic measure of the affine line $\mathbb{A}^1_\kappa$.

In this case let

$$p_{\lambda'_1,\lambda'_2} : \overline{\mathrm{N}}(E;\underline{\lambda}') \to \mathrm{Pic}_{\lambda'_1}(K)$$

be the projection. Let $0 = E_0' \subset E_1' \subset E_2' = E$ be a filtration of E, then $p_{\lambda_1',\lambda_2'}$ sends it to E_1' in the Picard group. If we fix the element E_1' in the Picard group, the preimages $p^{-1}_{\lambda_1',\lambda_2'}(E_1')$ contains all the diagrams:

$$\begin{array}{ccc} E_1' & \xrightarrow{\ \rho\ } & E \\ {\scriptstyle \phi_1}\downarrow & & \downarrow{\scriptstyle \phi} \\ E_1' \otimes \mathcal{O}_X(K) & \xrightarrow{\ \rho'\ } & E \otimes \mathcal{O}_X(K) \end{array}$$

modulo isomorphisms. Since once the inclusion ρ, ρ' and ϕ is fixed, then the Higgs field ϕ_1 is determined by the commutative diagram. Then this space is an affine bundle over the projective space $\mathbb{P}(\mathrm{Hom}(E_1', E))$ with rank $\mathrm{rk} = \dim(E, E \otimes \mathcal{O}_X(K))$. Thus the coefficient of $z_1^{\lambda_1'} z_2^{\lambda_2'}$ with $\lambda_1' + \lambda_2' = l = \deg(E)$ in $\overline{N}_E(z_1, z_2)$ is given by

$$\int_{E_1' \in \mathrm{Pic}_{\lambda_1'}(X)} \frac{\mathbb{L}^{\dim \mathrm{Hom}(E_1', E)} - 1}{\mathbb{L} - 1} \mathbb{L}^{\mathrm{rk}} d\mu_{E_1'}$$

and the coefficient at the same monomial in $(\mathbb{L} z_1/z_2)^{2-2g} \overline{N}_E(\mathbb{L} z_2, \mathbb{L}^{-1} z_1)$ is given by:

$$\int_{E_1' \in \mathrm{Pic}_{\lambda_2'+2-2g}(X)} \mathbb{L}^{\lambda_2' - \lambda_1' + 2 - 2g} \frac{\mathbb{L}^{\dim \mathrm{Hom}(E_1'', E)} - 1}{\mathbb{L} - 1} \mathbb{L}^{\mathrm{rk}} d\mu_{E_1''}$$

Now if we let

$$\sigma : \mathrm{Pic}_{\lambda_1'}(X) \to \mathrm{Pic}_{\lambda_2'+2-2g}(X); \quad E_1' \mapsto (E_1')^* \otimes \Lambda^2 E \otimes \omega_X^*,$$

the Riemann-Roch theorem tells us that

$$\dim \mathrm{Hom}(E_1', E) - \dim \mathrm{Hom}(E_1'', E) = \lambda_2' - \lambda_1' + 2 - 2g.$$

So the difference of the coefficients of $z_1^{\lambda_1'} z_2^{\lambda_2'}$ in the two side of $\overline{N}_E(z_1, z_2) = (z_1/z_2)^{2-2g} \overline{N}_E(z_2, z_1)$ is given by:

$$\int_{E_1' \in \mathrm{Pic}_{\lambda_1'}(X)} \frac{\mathbb{L}^{\lambda_2' - \lambda_1' + 2 - 2g} - 1}{\mathbb{L} - 1} \mathbb{L}^{\mathrm{rk}} d\mu_{E_1'} = \mu(\mathrm{Pic}_{\lambda_1'}(X)) \cdot \frac{\mathbb{L}^{\lambda_2' - \lambda_1' + 2 - 2g} - 1}{\mathbb{L} - 1} \mathbb{L}^{\mathrm{rk}}.$$

The Picard groups satisfy the condition $\mu(\mathrm{Pic}_{\lambda_1'}(X)) = \mu(\mathrm{Pic}_0(X))$ and then the difference between the two side of $\overline{N}_E(z_1, z_2) = (z_1/z_2)^{2-2g} \overline{N}_E(z_2, z_1)$ is given by:

$$\begin{aligned} & \frac{\mu(\mathrm{Pic}_0(X))}{\mathbb{L} - 1} \sum_{\lambda_1' + \lambda_2' = \deg(E)} (\mathbb{L}^{\lambda_2' - \lambda_1' + 2 - 2g} - 1) z_1^{\lambda_1'} z_2^{\lambda_2'} \cdot \mathbb{L}^{\mathrm{rk}} \\ & \quad = \frac{\mu(\mathrm{Pic}_0(X))}{\mathbb{L} - 1} \left(\mathbb{L}^{\mathrm{rk}} z_2^{\deg(E)} \mathbb{L}^{\deg(E) + 2 - 2g} \delta\left(\frac{z_1}{\mathbb{L}^2 z_2} \right) - \mathbb{L}^{\mathrm{rk}} z_2^{\deg(E)} \delta\left(\frac{z_1}{z_2} \right) \right) \end{aligned}$$

where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ is the Fourier series of the delta-function at 1. Then we apply [34, Lemma (1.3.4)] and letting $\mathbb{L} = 1$ the result follows. $\square$

5.3. **The case that the divisor $D = \mathbb{P}^1$.** In this section we study a special case that the self-intersection index of D in S, which is denoted by D_S^2, is a negative integer. Still let $S^0 = S \backslash D$, and fix a sheaf E^0 of rank r on S^0.

Let $\mathscr{M}^H(S;D)$ be the moduli stack of relative stable coherent sheaves E on S with Hilbert polynomial H such that

$$E|_{S^0} = E^0.$$

From Lemma 4.2, the functor is represented by an ind-scheme. But in the case that the self-intersection number D_S is negative, from [34, Theorem 2.21] $\mathscr{M}^H(S;D)$ is represented by a scheme.

Recall that $\widehat{S}_D$ is the formal completion of S along the smooth divisor D, which is a stft formal scheme over R with the underlying scheme X. From Theorem 4.1 $\widehat{\mathscr{M}}_R^H(\widehat{S}_D)$ is a formal scheme and $\widehat{\mathscr{M}}_R^H(\widehat{S}_D;D)$ is a formal subscheme of $\widehat{\mathscr{M}}_R^H(\widehat{S}_D)$.

Theorem 5.6. *The formal completion $\widehat{\mathscr{M}^H(S)}_{\mathscr{M}^H(S;D)}$ of $\mathscr{M}^H(S)$ along the subscheme $\mathscr{M}^H(S;D)$ is isomorphic to the formal scheme $\widehat{\mathscr{M}}_R^H(\widehat{S}_D;D)$.*

Proof. Let T be a scheme and $F \to T$ a family of stable torsion free sheaves with Hilbert polynomial H such that

$$F|_{S^0 \times T} \cong E^0_{S^0 \times D}.$$

Let $i : S^0 \hookrightarrow S$ be the inclusion. Then F must be some stable subsheaf in

$$i_* E|_{S^0 \times T} = i_* E^0.$$

Let $\mathfrak{m} \subset \mathcal{O}_S$ be the ideal sheaf of D in S. Then from [34, §2.2.2],

$$F \subset \mathrm{Quot}(\mathfrak{m}^{-N}/\mathfrak{m}^N F^0)$$

is a closed subscheme of the quot scheme for some $N >> 0$. And also in this case since the moduli space $\mathscr{M}^H(S;D)$ is scheme, and we can take $F \to T$ a family of stable torsion free sheave for $T \to \mathscr{M}^H(S;D)$. Taking the formal completion

$$\widehat{F} \to \widehat{T}$$

of the family gives an element in $\widehat{\mathscr{M}}_R^H(\widehat{S}_D;D)(\widehat{T})$. Then the result follows since from [28, Proposition 4.8], $\widehat{\mathscr{M}}_R^H(\widehat{S}_D) \cong \widehat{\mathscr{M}^H(S)}$. □

5.3.1. *Eisenstein series for the formal neighborhoods.* Let us write down the Eisenstein series for the pair $(\mathcal{S}, \mathcal{D})$, where $\mathcal{S}$ is the root stack $\mathcal{S} = \sqrt[d]{(S,D)}$ and $\mathcal{D} = p^{-1}(D)$. Note that we have made $d = r$, the rank of the coherent sheaves will be the same as the d.

Similar proof as in [34, §2.2.2] shows that we have a moduli scheme

$$\mathscr{M}^H(\mathcal{S};\mathcal{D})$$

of relative stable coherent torsion free sheaves E on $\mathcal{S}$ with Hilbert polynomial H ad generating sheaf Ξ such that $E|_{\mathcal{S}^0} = E^0$, a fixed stable sheaf.

Let $\widehat{\mathcal{S}}_{\mathcal{D}}$ be the formal completion of $\mathcal{S}$ along $\mathcal{X}$. We have the formal moduli space

$$\widehat{\mathscr{M}}_R^H(\widehat{\mathcal{S}}_{\mathcal{D}};\mathcal{D})$$

of stable sheaves $\widehat{E}$ on $\widehat{\mathcal{S}}_{\mathcal{D}}$ with fixed polynomial H and the corresponding generating sheaf Ξ such that $\widehat{E}|_{\widehat{\mathcal{S}}^0_{\mathcal{D}}} \cong \widehat{E}^0$. Then Theorem 5.6 gives:

Proposition 5.7.
$$\widehat{\mathscr{M}}^H_R(\widehat{\mathcal{S}}_{\mathcal{D}};\mathcal{X}) \cong \widehat{\mathscr{M}}^H(\mathcal{S})_{\mathscr{M}^H(\mathcal{S};\mathcal{D})},$$
the formal completion of $\mathscr{M}^H(\mathcal{S})$ along $\mathscr{M}^H(\mathcal{S};\mathcal{D})$.

We have similar results for the moduli of Higgs pairs. Let $\widehat{\mathcal{N}}^H_R(\widehat{\mathcal{S}}_{\mathcal{D}};\mathcal{D})$ be the formal moduli scheme of relative formal Higgs pairs
$$(\widehat{E},\phi)$$
such that $\widehat{E}|_{\widehat{\mathcal{S}}^0_{\mathcal{D}}} = \widehat{E}^0$ and $\phi : \widehat{E} \to \widehat{E} \otimes K_{\widehat{\mathcal{S}}_{\mathcal{D}}}$ is the Higgs field.

Also let $\mathcal{N}^H(\mathcal{S};\mathcal{D})$ be the moduli stack of relative Higgs pairs (E,ϕ) such that E is a stable torsion free sheaf on $\mathcal{S}$ with Hilbert polynomial H, $E|_{\mathcal{S}^0} = E^0$ and $\phi : E|_{\mathcal{S}^0} \to E|_{\mathcal{S}^0} \otimes K_{\mathcal{S}^0}$ is a section. Then we have:

Proposition 5.8. *The formal scheme $\widehat{\mathcal{N}}^H_R(\widehat{\mathcal{S}}_{\mathcal{D}};\mathcal{D})$ is isomorphic to the formal completion of $\mathcal{N}^H(\mathcal{S})$ along the subscheme $\mathcal{N}^H(\mathcal{S};\mathcal{D})$.* $\square$

For a fixed Hilbert polynomial $H \in \mathbb{Q}[m]$ corresponding to a generating sheaf Ξ on $\mathcal{S}$, we take H coming from a K-group class $\mathbf{c} \in K_0(\mathcal{S}) \cong K_0(\widehat{\mathcal{S}})$ and write the formal moduli scheme $\widehat{\mathscr{M}}^H_R(\widehat{\mathcal{S}}_{\mathcal{D}};\mathcal{D})$, $\widehat{\mathcal{N}}^H_R(\widehat{\mathcal{S}}_{\mathcal{D}};\mathcal{D})$ as $\widehat{\mathscr{M}}^{\mathbf{c}}_R(\widehat{\mathcal{S}}_{\mathcal{D}};\mathcal{D})$, $\widehat{\mathcal{N}}^{\mathbf{c}}_R(\widehat{\mathcal{S}}_{\mathcal{D}};\mathcal{D})$ repectively.

Let us explain the generating functions. We fix the filtration
$$F_0K_0(\mathcal{S}) \subset F_1K_0(\mathcal{S}) \subset F_2K_0(\mathcal{S})$$
where $F_iK_0(\mathcal{S})$ is the subgroup of $K_0(\mathcal{S})$ such that the support of the elements in $F_iK_0(\mathcal{S})$ has dimension $\leqslant i$. The orbifold Chern character morphism is defined by:

$$\widetilde{\mathrm{Ch}} : K_0(\mathcal{S}) \to H^*_{\mathrm{CR}}(\mathcal{S},\mathbb{Q}) = H^*(I\mathcal{S},\mathbb{Q}) \tag{5.3.1}$$

where $H^*_{\mathrm{CR}}(\mathcal{S},\mathbb{Q})$ is the Chen-Ruan cohomology of $\mathcal{S}$. The inertia stack
$$I\mathcal{S} = \mathcal{S}\bigsqcup \sqcup_{i=1}^{r-1}\mathcal{C}_i$$
where each $\mathcal{C}_i = \mathcal{C}$ is the stacky divisor of $\mathcal{S}$. We should understand that the inertia stack is indexed by the element $g \in \mu_r$, $\mathcal{S}_g \cong \mathcal{C}$ is the component corresponding to g. It is clear that $\mathcal{S}_1 = \mathcal{S}$ and $\mathcal{S}_g = \mathcal{C}$ if $g \neq 1$. Let $\zeta \in \mu_r$ be the generator of μ_r. Then
$$H^*(I\mathcal{S},\mathbb{Q}) = H^*(\mathcal{S})\bigoplus \oplus_{i=1}^{r-1}H^*(\mathcal{C}_i),$$
where $\mathcal{C}_i$ corresponds to the element ζ^i. The cohomology of $H^*(\mathcal{C}_i)$ is isomorphic to $H^*(C)$. For any coherent sheaf E, the restriction of E to every $\mathcal{C}_i$ has a μ_r-action such that it acts by $e^{2\pi i\frac{f_i}{r}}$, and we let

$$\widetilde{\mathrm{Ch}}(E) = (\mathrm{Ch}(E), \oplus_{i=1}^{r-1}\mathrm{Ch}(E|_{\mathcal{C}_i})), \tag{5.3.2}$$

where
$$\mathrm{Ch}(E) = (\mathrm{rk}(E), c_1(E), c_2(E)) \in H^*(\mathcal{S}),$$
and
$$\mathrm{Ch}(E|_{\mathcal{C}_i}) = \left(e^{2\pi i\frac{f_i}{r}}\,\mathrm{rk}(E), e^{2\pi i\frac{f_i}{r}}c_1(E|_{\mathcal{C}_i}\right) \in H^*(\mathcal{C}).$$

In order to write down the generating function later. We introduce some notations. We roughly write

$$\widetilde{\mathrm{Ch}}(E) = (\widetilde{\mathrm{Ch}}_g(E))$$

where $\widetilde{\mathrm{Ch}}_g(E)$ is the component in $H^*(\mathcal{S}_g)$ as in (5.3.2). Then define:

$$\left(\widetilde{\mathrm{Ch}}_g\right)^k := \left(\widetilde{\mathrm{Ch}}_g\right)_{\dim \mathcal{S}_g - k} \in H^{\dim \mathcal{S}_g - k}(\mathcal{S}_g). \tag{5.3.3}$$

The k is called the codegree. In our inertia stack $\mathcal{S}_g$ is either the whole $\mathcal{S}$, or $\mathcal{C}$, therefore if we have a rank 2 $\mathbb{C}^*$-fixed Higgs pair (E,ϕ) with fixed $c_1(E) = -c_1(\mathcal{S})$, then $\left(\widetilde{\mathrm{Ch}}_g\right)^2(E) = 2$, the rank; while

$$\left(\widetilde{\mathrm{Ch}}_g\right)^1(E) = \begin{cases} -c_1(\mathcal{S}), & g = 1; \\ 2e^{2\pi i \frac{f_i}{r}}, & g = \zeta^i \neq 1. \end{cases}$$

Also we have

$$\left(\widetilde{\mathrm{Ch}}_g\right)^0(E) = \begin{cases} c_2(E), & g = 1; \\ e^{2\pi i \frac{f_i}{r}} c_1(E|_{\mathcal{C}_i}), & g = \zeta^i \neq 1. \end{cases}$$

We introduce variables q to keep track of the second Chern class $n = c_2(E)$ of the torsion free sheaf E, $q_1, \cdots, q_{r-1}$ to keep track of the classes $n_i = c_1(E|_{\mathcal{C}_i})$ for $i = 1, \cdots, r-1$. Let $\mathbf{q} = (q, q_1, \cdots, q_{r-1})$.

Definition 5.9. *We define:*

$$E^{\mathrm{for}}(q) = E_{\widehat{\mathcal{S}}_{\mathcal{D}}, \widehat{E}^0}(q) = \sum_{\mathbf{c} \in K_0(\mathcal{S})} \chi(\widehat{\mathscr{M}}^{\mathbf{c}}_R(\widehat{\mathcal{S}}_{\mathcal{D}}; \mathcal{D}))\mathbf{q}^{\mathbf{c}}$$

and

$$F^{\mathrm{for}}(q) = F_{\widehat{\mathcal{S}}_{\mathcal{D}}, \widehat{E}^0}(q) = \sum_{\mathbf{c} \in K_0(\mathcal{S})} \chi(\widehat{\mathcal{N}}^{\mathbf{c}}_R(\widehat{\mathcal{S}}_{\mathcal{D}}; \mathcal{D}))\mathbf{q}^{\mathbf{c}}$$

where for the formal schemes, the Euler characteristic $\chi(\widehat{\mathscr{M}}^{\mathbf{c}}_R(\widehat{\mathcal{S}}_{\mathcal{D}}; \mathcal{D}))$ and $\chi(\widehat{\mathcal{N}}^{\mathbf{c}}_R(\widehat{\mathcal{S}}_{\mathcal{D}}; \mathcal{D}))$ are defined using étale cohomology. Here $\mathbf{q}^{\mathbf{c}} = q^n \cdot q_1^{n_1} \cdots q_{r-1}^{n_{r-1}}$.

Let us go back to the case that the pair is (S, D) for smooth projective surface S and $D \subset S$ a smooth divisor. Then we consider $q_1, q_2, \cdots, q_{r-1}$ will count the parabolic degree of the rank r torsion free sheaves restricted to D.

Let us prove a functional equation at rank $r = 2$ case.

Proposition 5.10. ***(Functional equation)*** *In the case of rank $r = 2$, the series $E^{\mathrm{for}}(q)$ represents a rational function with only poles being simple poles along $q_1 = q_2$. It satisfies the functional equation:*

$$E^{\mathrm{for}}(q, q_1, q_2) = (q_1/q_2)^{2-2g} E^{\mathrm{for}}(q, q_2, q_1).$$

Proof. From the definition of the moduli space $\widehat{\mathscr{M}}^{\mathbf{c}}_R(\widehat{\mathcal{S}}_{\mathcal{D}}; \mathcal{D})$, any formal sheaf on $\widehat{\mathcal{S}}_{\mathcal{D}}$ is supported on the curve $\mathcal{D}$. So the series $E^{\mathrm{for}}(q)$ is just $\overline{N}(E; \underline{\lambda}')$. Hence from Proposition 5.2.1, $\overline{N}_E(z)$ satisfies the function equation, and $E^{\mathrm{for}}(q, q_1, q_2)$ also satisfies the functional equation. □

5.3.2. *Blow-up formula for Vafa-Witten invariants.* In this section we prove a blow-up formula for the Vafa-Witten invariants vw in [48], [49]. We also talk about the stacky case defined in [25]. Note that in [15], Goeschett-Kool proved a blow-up formula for virtual Euler numbers.

Let S be a projective surface. Recall that in [48, §6], fixing a Hilbert polynomial of Gieseker stable Higgs pairs is equivalent to fixing the topological data $(r, c_1, c_2) \in H^*(S, \mathbb{Z})$, with $r > 0$. Then the Vafa-Witten invariants are given by:

$$\mathrm{VW}_{r,c_1,c_2}(S) = \int_{[\mathcal{N}^{\perp}_{r,c_1,c_2}(S)^{\mathbb{C}^*}]^{\mathrm{vir}}} \frac{1}{e(N^{\mathrm{vir}})}$$

where $\mathcal{N}^{\perp}_L$ is the moduli space of stable Higgs pairs (E, ϕ) with $\det(E) = L$, ϕ is trace-free and $c_1(L) = c_1$. The moduli space $\mathcal{N}^{\perp}_L$ admits a symmetric obstruction theory and a $\mathbb{C}^*$-action by scaling the fibres. The invariants above are defined by virtual localization of Graber-Pandharipande [12].

As recalled in §4.1, we are mainly interested in the second Vafa-Witten invariants vw which are defined by

$$\mathrm{vw}_{r,c_1,c_2}(S) = \chi(\mathcal{N}^{\perp}_L, \nu_{\mathcal{N}})$$

where $\nu_{\mathcal{N}}$ is the Behrend function and $\chi(\mathcal{N}^{\perp}_L, \nu_{\mathcal{N}})$ is the weighted Euler characteristic weighted by the Behrend function in [2]. The moduli space $\mathcal{N}^{\perp}_L$ is non-compact, but there exists a $\mathbb{C}^*$-action on $\mathcal{N}^{\perp}_L$ by scaling the Higgs fields. We have:

$$\mathrm{vw}_{r,c_1,c_2}(S) = \chi\left(\left(\mathcal{N}^{\perp}_L\right)^{\mathbb{C}^*}, \nu_{\mathcal{N}}|_{(\mathcal{N}^{\perp}_L)^{\mathbb{C}^*}}\right).$$

The $\mathbb{C}^*$-action on $\mathcal{N}^{\perp}_L$ has two type of fixed components, see [48, §7], and [25, §3.5]:

(1) The Higgs field $\phi = 0$. Then in this case the fixed locus $\mathscr{M}^{(1)}$ is isomorphic to the moduli space of stable sheaves on $\mathcal{S}$ with determinant L. One can think of the moduli space $\mathcal{N}^{\perp}_L$ over such locus as the dual of the obstruction sheaf $\mathrm{Ob}_{\mathscr{M}^{(1)}}$, see [22]. Hence from the main results in [22],

$$\mathrm{vw}_{r,c_1,c_2}(S) = \chi\left(\left(\mathcal{N}^{\perp}_L\right)^{\mathbb{C}^*}, \nu_{\mathcal{N}}|_{(\mathcal{N}^{\perp}_L)^{\mathbb{C}^*}}\right) = (-1)^{\mathrm{vd}} \chi(\mathscr{M}^{(1)}), \tag{5.3.4}$$

which is the signed Euler characteristic of $\mathscr{M}^{(1)}$. If the projective surface satisfies $K_S \leqslant 0$, then the only $\mathbb{C}^*$-fixed Higgs pairs have $\phi = 0$, and

$$\mathrm{VW}_{r,c_1,c_2}(S) = \mathrm{vw}_{r,c_1,c_2}(S) = (-1)^{\mathrm{vd}} \chi(\mathscr{M}^{(1)}).$$

In [49], [42], the same result is proved for the counting invariants of semistable Higgs pairs. We prove a blow-up formula for the Vafa-Witten invariants in this case, and the blow-up formula is really coming from the blow-up formula for Donaldson-Thomas type invariants for the projective surface S as proved in [40], [34].

(2) The second type of fixed locus $\mathscr{M}^{(2)}$ corresponds to the case that in the Higgs pair the Higgs field $\phi \neq 0$. In this case, as proved in [48], [25], $\mathscr{M}^{(2)}$ is isomorphic to the union of nested Hilbert schemes.

It is interesting to see how the Vafa-Witten invariants contributed from this component are related to the Eisenstein series of the Higgs pairs on the curve X.

5.3.3. *Eisenetein series for $D = \mathbb{P}^1$.* In this section we let S be a smooth projective surface, and $D = \mathbb{P}^1 \subset S$ such that the self-intersection number $D_S^2 = -d$. The formal completion $\widehat{S}_D = \varinjlim_n (S/\mathcal{I}_D^n)$ of S along D is the one along the ideal sheaf $\mathcal{I}_D$ of D inside S. It is isomorphic to the formal completion of the normal bundle $N_{D/S} = \mathcal{O}_D(-d)$ along D. For such a formal scheme, the formal moduli space $\widehat{\mathscr{M}}_R^H(\widehat{S}_D; D)$ of stable torsion free coherent sheaves $\widehat{E}$ on $\widehat{S}_D$ such that

$$\widehat{E}|_{S^0} \cong \widehat{E}^0$$

is isomorphic to the formal completion of the moduli space $\mathscr{M}^H(S)$ along $\mathscr{M}^H(S; D)$, the relative moduli scheme as in [34, §2.2.1]. The Hilbert polynomial H is determined by topological data

$$(r, n, a)$$

and we have the generating series

$$E(q) = E_{\widehat{\mathcal{S}}_{\mathcal{D}}, \widehat{E}^0}(q) = \sum_{n,a} \chi(\widehat{\mathscr{M}}_R^n(\widehat{\mathcal{S}}_{\mathcal{D}}; \mathcal{D}))q^n$$

where $\widehat{\mathscr{M}}_R^{n,a}(\widehat{\mathcal{S}}_{\mathcal{D}}; \mathcal{D})$ is the formal moduli space of stable relative sheaves on $\widehat{S}_X$ with topological data (r, n, a) and

$$\widehat{E}|_{\widehat{S}_D^0} \cong \widehat{E}^0.$$

The formal moduli scheme $\widehat{\mathscr{M}}_R^{n,a}(\widehat{\mathcal{S}}_{\mathcal{D}}; \mathcal{D})$ is isomorphic to the formal completion $\widehat{\Gamma}_{n,a}(\mathcal{Q})$ over $\mathrm{Spf}(R)$ as in [34, Theorem 6.1.1]. In the case $D = \mathbb{P}^1$, [34, §7] calculated the Eisenstein series $E(q)$. Note that in [34, §7], the author did the calculation for any reductive group G, and the motivic Eisenstein series. We only need the Euler characteristic version in this paper, and leave the motivic version of the Eisenstein series for the future research on the motivic Vafa-Witten invariants.

Proposition 5.11. ([34, Theorem 7.4.6]) *In the case that $D = \mathbb{P}^1$, and $D_S^2 = -1$, we have*

$$E_{\widehat{\mathcal{S}}_{\mathcal{D}}, \widehat{E}^0}(q) = E^{\mathrm{for}}(q) = \sum_{a \in \mathbb{Z}} q^{-\frac{a^2}{2}}.$$

Let $P \in S$ be a point and $\sigma : \widetilde{S} \to S$ be the blow-up of S along the point P. Let $D := \pi^{-1}(P)$ be the exceptional divisor. Let H be an ample divisor on S and $\mathscr{M}_{S,n}^H$ the moduli space of H-stable coherent sheaves on S with $c_2(E) = n$. Consider the divisor

$$H_i = i \cdot \sigma^* H - aX$$

on $\widetilde{S}$ for $a = 0, 1$. For $i >> 0$, H_i is ample and denote by $\mathscr{M}^{H_\infty}(\widetilde{S}, n)$ the corresponding moduli space of stable coherent sheaves. Then Qin-Li [40] proved a blow-up formula:

Proposition 5.12.

$$\sum_n \chi(\mathscr{M}^{H_\infty}(\widetilde{S}, n))q^n = \frac{E_{\widehat{\mathcal{S}}_{\mathcal{D}}, \widehat{E}^0}(q)}{\left(\prod_{m \geqslant 0}(1-q)^m\right)^2} \cdot \sum_n \chi(\mathscr{M}^H(S, n))q^n. \tag{5.3.5}$$

Proof. The case that the moduli space only contains stable vector bundles is calculated in [34, Theorem 7.4.6].

To get the general formula of Qin-Li [40], we first analyze the moduli space $\widehat{\mathscr{M}}_R^{\mathbf{c}}(\widehat{\mathcal{S}}_{\mathcal{D}};\mathcal{D})$. Let F be a stable torsion free sheaf on $\widehat{\mathscr{M}}_R^{\mathbf{c}}(\widehat{\mathcal{S}}_{\mathcal{D}};\mathcal{D})$. Since the formal scheme $\widehat{\mathscr{M}}_R^{\mathbf{c}}(\widehat{\mathcal{S}}_{\mathcal{D}};\mathcal{D})$ is a formal completion of the relative moduli space $\mathscr{M}^{\mathbf{c}}(S;D)$, we can take F as a relative sheaf on S such that $F|_{S^0}$ is trivial. Look at the exact sequence

$$0 \to F|_D \to F \to F/F|_D \to 0,$$

where the torsion sheaf $F|_D$ is supported on D, which is a rank two vector bundle on D. The quotient $F/F|_D$ is a rank two coherent sheaf with topological invariants $\mathbf{c}_0$ Hence this will be an ideal sheaf of points on $N_{D/S}$ of length $n = c_2(F)$. So the Euler characteristic of the moduli space $\mathscr{M}^{\mathbf{c}}(S;D)$ is the same as the product of the Euler characteristic of $\widehat{\mathscr{M}}_R^{\mathbf{c}}(\widehat{\mathcal{S}}_{\mathcal{D}};\mathcal{D})$ with the Euler characteristic of the Hilbert scheme of points on $N_{D/S}$. Hence

$$\sum_n \chi(\mathscr{M}^{\mathbf{c}}(S;D))q^n = \frac{E^{\mathrm{for}}(q,q_1,q_2)}{\left(\prod_{m\geqslant 0}(1-q)^m\right)^2}$$

Since

$$\sum_n \chi(\mathscr{M}^{H_\infty}(\widetilde{S},n))q^n = \sum_n \chi(\mathscr{M}^{\mathbf{c}}(S;D))q^n \cdot \sum_n \chi(\mathscr{M}^{H}(S,n))q^n,$$

we are done. □

We have a similar blow-up formula for the Vafa-Witten invariants:

Theorem 5.13. *Let $\sigma : \widetilde{S} \to S$ be the blow-up of the surface S along a point $P \in S$. Let*

$$\mathrm{vw}^{\widetilde{S}}_{\widetilde{c}_1,\widetilde{c}_2} = \chi(\mathcal{N}_L^{\perp}(\widetilde{S}),\nu_{\mathcal{N}_L^{\perp}})$$
$$\mathrm{vw}^{S}_{c_1,c_2} = \chi(\mathcal{N}_L^{\perp}(S),\nu_{\mathcal{N}_L^{\perp}})$$

be the small Vafa-Witten invariants of $\widetilde{S}, S$ respectively with topological data $\widetilde{c}_1, c_1, \widetilde{c}_2 = c_2$. Then we have:

$$\sum_n \mathrm{vw}^{\widetilde{S}}_{\widetilde{c}_1,n}\, q^n = E_{\widehat{\mathcal{S}}_{\mathcal{D}},\widehat{E}^0}(q)\cdot \sum_n \mathrm{vw}^{S}_{c_1,n}\, q^n.$$

Proof. Since both $\widetilde{S}$ and S satisfy $K_{\widetilde{S}}, K_S \leqslant 0$, the Vafa-Witten invariants satisfy

$$\mathrm{vw}^{\widetilde{S}}_{\widetilde{c}_1,n} = (-1)^{\mathrm{vd}^{\widetilde{S}}}\cdot \chi(\mathscr{M}^{H_\infty}_{\widetilde{c}_1,n}(\widetilde{S}))$$

and

$$\mathrm{vw}^{S}_{c_1,n} = (-1)^{\mathrm{vd}^{S}}\cdot \chi(\mathscr{M}^{H}_{c_1,n}(S)),$$

where $\mathrm{vd}^{\widetilde{S}}, \mathrm{vd}^S$ are the virtual dimension of the moduli spaces $\mathscr{M}^{H_\infty}_{\widetilde{c}_1,n}(\widetilde{S}))$ and $\mathscr{M}^{H}_{c_1,n}(S)$ respectively.

Let us calculate the virtual dimensions:

$$\mathrm{vd}^{\widetilde{S}} = 2r\widetilde{c}_2 - (r-1)\widetilde{c}_1^2 - (r^2-1)\chi(\mathcal{O}_{\widetilde{S}})$$
$$\mathrm{vd}^{S} = 2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(\mathcal{O}_{S}).$$

We calculate

$$\chi(\mathcal{O}_{\widetilde{S}}) = \frac{1}{12}(K^2_{\widetilde{S}} + \chi_{\mathrm{top}}(\widetilde{S}))$$

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K_S^2 + \chi_{\mathrm{top}}(S))$$

and

$$\widetilde{c}_2 = c_2 = n; \quad \widetilde{c}_1 = \sigma^* c_1 - aD$$

for $a = 0$ or 1. Also we have

$$K_{\widetilde{S}} = \sigma^* K_S + D,$$

and

$$K_{\widetilde{S}}^2 = K_S^2 - 1; \chi_{\mathrm{top}}(S) + 1 = \chi_{\mathrm{top}}(\widetilde{S}).$$

Then we calculate:

$$\mathrm{vd}^{\widetilde{S}} - \mathrm{vd}^S = \frac{1}{12}(r^2-1)(K_S^2 + \chi_{\mathrm{top}}(S) - K_S^2 + 1 - \chi_{\mathrm{top}}(S) - 1) = 0.$$

So let

$$A := -(r-1)c_1^2 - (r^2-1)\chi(\mathcal{O}_S) = -(r-1)\widetilde{c}_1^2 - (r^2-1)\chi(\mathcal{O}_{\widetilde{S}}),$$

we have:

$$\begin{aligned}
\sum_n \mathrm{vw}_{\widetilde{c}_1,n}^{\widetilde{S}}\, q^n &= \sum_n (-1)^A \cdot \chi(\mathscr{M}_{\widetilde{c}_1,n}^{\widetilde{S}}) q^n \\
&= (-1)^A \cdot \sum_n \chi(\mathscr{M}_{\widetilde{c}_1,n}^{\widetilde{S}}) q^n \\
&= (-1)^A \cdot \frac{E_{\widehat{\mathcal{S}}_{\mathcal{D}},\widehat{E}^0}(q)}{\left(\prod_{m\geqslant 0}(1-q)^m\right)^2} \cdot \sum_n \chi(\mathscr{M}_{c_1,n}^S) q^n \\
&= \frac{E_{\widehat{\mathcal{S}}_{\mathcal{D}},\widehat{E}^0}(q)}{\left(\prod_{m\geqslant 0}(1-q)^m\right)^2} \cdot \sum_n \mathrm{vw}_{c_1,n}^S\, q^n.
\end{aligned}$$

□

Remark 5.14. *(1) In the cases that $K_{\widetilde{S}}, K_S \leqslant 0$, the Vafa-Witten invariants*

$$\mathrm{VW}_{\widetilde{c}_1,n}^{\widetilde{S}} = \mathrm{vw}_{\widetilde{c}_1,n}^{\widetilde{S}}$$

and

$$\mathrm{VW}_{c_1,n}^S = \mathrm{vw}_{c_1,n}^S\,.$$

Hence we also get the blow-up formula for the Vafa-Witten invariants in [48].

(2) It is more interesting to calculate the Vafa-Witten invariants coming from the type two fixed locus $\mathscr{M}^{(2)}$ of $\mathcal{N}_L^\perp$ under the $\mathbb{C}^*$-action, which is called the monopole *branch of the Vafa-Witten invariants. A blow-up formula for such invariants is more interesting and it is hoped that this is related to the generalized Higgs pairs on curves.*

In [15]*, Göttsche and Kool made a conjecture for the monopole branch in $\mathcal{N}_L^\perp$ by the S-duality modular transformation $\tau \to -\tau^{-1}$. It is interesting to see what properties this implies the geometric Eisenstein series.*

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