COUNTING LOG SURFACES OF GENERAL TYPE VIA MODULI SPACE OF STABLE MAPS I: MODULI SPACES

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ABSTRACT. This is the first paper in a project aiming to study the enumerative counting invariants for log surfaces of general type via moduli space of stable maps. The current paper focuses on the construction of the moduli stack of stable maps from certain stacky log surfaces of general type to a target projective Deligne-Mumford stack.

CONTENTS

1. INTRODUCTION

We work over the field of complex numbers. The main goal of this paper is to construct the moduli stack of stable maps from certain stacky semi-log-canonical (slc) log varieties of general type to a projective Deligne-Mumford stack. In a sequel we will use the techniques in [38] to construct a virtual fundamental class on certain moduli spaces of stable maps constructed in this paper and define Gromov-Witten counting surface invariants.

1.1. **Motivation.** Curve counting theory in modern enumerative algebraic geometry has been a very important research topic for almost thirty years. The Gromov-Witten (GW) invariants of counting curves are defined by the virtual fundamental class of Kontsevich's moduli space of stable maps. Let W be a smooth projective scheme, and $\overline{M}_{g,n}(W,\beta)$ be the moduli space of stable maps from genus g, n-marked curves to W with class $\beta \in H_2(W,\mathbb{Z})$. The moduli space $\overline{M}_{g,n}(W,\beta)$ admits a perfect obstruction theory in the sense of Behrend-Fantechi [14], and Li-Tian [53]. The perfect obstruction theory induces a virtual fundamental class

$$[\overline{M}_{g,n}(W,\beta)]^{\mathrm{vir}} \in H_{2\,\mathrm{vd}}(\overline{M}_{g,n}(W,\beta),\mathbb{Q}),$$

where $\mathrm{vd} = (\dim(W) - 3)(1 - g) + \int_\beta c_1(T_W) + n$ is the virtual dimension. The Gromov-Witten invariants for W are defined by integrating cohomology classes on the moduli space over the virtual fundamental class; see [13]. The Gromov-Witten invariants are deformation invariant, and satisfy many interesting properties; see [53], [13].

Motivated by gauge theory in higher dimensions [24], in the case of projective threefold or Calabi-Yau threefold W, R. Thomas [72] defined the curve counting theory in W using the moduli space of ideal sheaves of the associated curves, called the Donaldson-Thomas (DT) invariants. The famous MNOP conjecture, due to [54], [55], stated that these two curve counting invariants are equivalent in terms of generating functions after change of variables which is called the GW/DT-correspondence. In [64], Pandharipande-Thomas developed the theory of moduli space of stable pairs $[\mathcal{O}_W \to F]$ for projective threefold W to count curves, where F is a pure one dimensional sheaf supported on a curve. The invariants are called Pandharipande-Thomas (PT) invariants. Both the DT and the PT invariants on a Calabi-Yau threefold W are motivic invariants proved by Behrend [12]. The DT/PT-correspondence conjectured that the DT invariants and PT invariants are related by wall crossing, which was proved by Bridgeland [16], and Toda [73] using the Hall algebras, Bridgeland stability conditions and Behrend function techniques.

In the case of Calabi-Yau fourfold W, recently Oh-Thomas [60] constructed an algebraic virtual fundamental class on the moduli space of stable sheaves on W, using the symmetric obstruction theory of the moduli space and a symmetry on the obstruction space. This virtual fundamental class is the same as Borisov-Joyce [15] using the analytic method; see [61]. Already there are a lot of work studying the curve counting theory in a Calabi-Yau fourfold W using DT and PT theory of the 4-fold W, see [18], [19], etc.

The counting surface theory in the next step in a Calabi-Yau 4-fold W is a very interesting direction. In physics literature of M-/F-theory by Gukov-Vafa-Witten [28], and SUSY Yang-Mills theory by Nekrasov [56], [57], the counting surface theory is seen to be a research direction in physics. In mathematics, in [26], [11], the authors have initiated the program to study the surface counting theory on a Calabi-Yau 4-fold W using Pandharipande-Thomas's theory of stable pairs $[\mathcal{O}_W \xrightarrow{s} F]$, where F is a two dimensional sheaf on W. In particular, in [11], Bae-Kool-Park defined the reduced virtual fundamental class $[M]^{\mathrm{red}}$ on the moduli space M of stable pairs $[\mathcal{O}_W \xrightarrow{s} F]$ in two cases: one is called PT_0-invariant which requires that the cokernel Q of s has dimension ≤ 0; the other is called PT_1-invariant which requires that the cokernel Q of s has dimension ≤ 1. Note that only the case PT_1 was defined in [26]. The reduced virtual fundamental class $[M]^{\mathrm{red}}$ can be applied to count surface pairs $(\mathbb{P}^2, D)$, where D is a divisor in $\mathbb{P}^2$, while in this case the virtual fundamental class $[M]^{\mathrm{vir}}$ in [26] is zero. Let Q be the cokernel of s. For PT_0, the two dimensional sheaf F is not necessarily pure and Q must be zero dimensional. This forces the structure of the underlying surface associated with F is very difficult to control and it may have very bad singularities. For PT_1, the two dimensional sheaf F is pure, and Q is a one dimensional sheaf. This forces the underlying surface associated with F satisfies Serre's condition S_1. Even in this case the structure of the surface is still very complicated.

The goal of our project is to count surfaces using stable maps to W. We only count semi-log-canonical (s.l.c.) log surfaces pairs (X, D) such that $K_X + D$ is g-very ample, where $g : (X, D) \to W$ is the stable map. Semi-log-canonical singularities of (X, D) are classified in [44, Theorem 4.23, Theorem 4.24] and [48, Chapter 5]. For instance, an s.l.c. surface X must be reduced, Cohen-Macaulay, and satisfies Serre's condition S_2. Thus the theory of counting surfaces using stable maps really gives much more restrictions on the structure of the underlying surfaces than the stable pair theory in [11], [26]. We mainly focus on the construction of moduli stack of stable maps in this paper.

1.2. **Moduli space of stable maps.** Let X be a projective surface with only s.l.c. singularities. An s.l.c. log pair (X, D) is an s.l.c. projective surface X, together with a $\mathbb{Q}$-divisor $D = \sum_i a_i D_i$ such that (X, D) only has s.l.c. singularities and D does not meet with the generic codimension one singular locus of X.

Let (X, D, x) be an s.l.c. surface germ. The index of $x \in X$ is, by definition, the least integer $r > 0$ such that $\omega_X^{[r]} = (\omega_{\mathcal{X}}^{\otimes r})^{\vee\vee}$ is invertible around x. Fixing an isomorphism $\theta : \omega_X^{[r]} \to \mathcal{O}_X$, each semi-log-canonical germ (X, D, x) defines a local cover $Z := \mathrm{Spec}_{\mathcal{O}_X}(\mathcal{O}_X \oplus \omega_X^{[1]} \oplus \cdots \oplus \omega_X^{[r-1]}) \to X$ under the $\mathbb{Z}_r$-action, where the multiplication is given by the isomorphism θ. The surface Z is Gorenstein, which implies that ω_Z is invertible. This cover is uniquely determined by the étale topology which we call the index one cover. All of these data of index one covers for s.l.c. germs (which locally gives the stack $[Z/\mathbb{Z}_r]$) glue to define a Deligne-Mumford stack $\pi : \mathfrak{X} \to X$ which is called the index one covering Deligne-Mumford stack. The dualizing sheaf $\omega_{\mathfrak{X}}$, which is étale locally given by $\omega_{[Z/\mathbb{Z}_r]}$, is invertible.

For the singularity germ (X, D, x), a deformation $(\mathcal{X}, \mathcal{D})/T$ over a scheme T is called $\mathbb{Q}$-Gorenstein if locally there is a $\mathbb{Z}_r$-equivariant deformation $(\mathcal{Z}, \widetilde{\mathcal{D}})/T$ of Z whose quotient is $(\mathcal{X}, \mathcal{D})/T$. Let $\omega_{\mathcal{X}/T}$ be the relative dualizing sheaf of $\mathcal{X}/T$. We define $\omega_{\mathcal{X}/T}^{[r]} := (\omega_{\mathcal{X}/T}^{\otimes r})^{\vee\vee} = i_*\omega_{\mathcal{X}^0/T}^{\otimes r}$, where $i : \mathcal{X}^0 \hookrightarrow \mathcal{S}$ is the inclusion of the Gorenstein locus of $\mathcal{X}/T$, which is the locus where $\omega_{\mathcal{X}/T}$ is invertible; see [29, §3.1] and [44, §5.4]. From Hacking [29, §3.2], let $\mathcal{X}/T$ be a $\mathbb{Q}$-Gorenstein deformation family of s.l.c. surfaces and $x \in \mathcal{X}$ has index r, then $\mathcal{Z}$ is given by $\mathcal{Z} := \mathrm{Spec}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}} \oplus \omega_{\mathcal{X}/T}^{[1]} \oplus \cdots \oplus \omega_{\mathcal{X}/T}^{[r-1]})$, where the multiplication is given by fixing a trivialization of $\omega_{\mathcal{X}/T}^{[r]}$ at the point x. The canonical covering $\mathcal{Z}$ of $x \in \mathcal{X}/T$ is uniquely determined by the étale topology. These data of local quotient stacks $[\mathcal{Z}/\mathbb{Z}_r]$ glue to give the index one covering Deligne-Mumford stack $\mathfrak{X}/T$ which is a flat family over T from [29, Lemma 3.5]. Therefore, let $\mathfrak{D} := \pi^{-1}(\mathcal{D})$, and we get a family $(\mathfrak{X}, \mathfrak{D})/T$, where $\pi : \mathfrak{X} \to \mathcal{X}$ is the map to its coarse moduli space.

An s.l.c. log surface pair (X, D) is called stable if its $K_X + D$ is ample. Let W be a smooth projective scheme. From [3], a map $g : (X, D) \to W$ is a stable map if the following conditions are satisfied:

(1) $D = \sum_{i=1}^n a_i D_i$ is a $\mathbb{Q}$-divisor of X;
(2) the pair (X, D) is s.l.c.;
(3) the divisor $K_X + D$ is relatively g-ample;
(4) let $H := g^*\mathcal{O}_W(1)$. There exists an $N > 0$ such that the sheaf $L_N := \mathcal{O}(N(K_X + D + 5H))$ is a line bundle.

Let T be a finite type scheme over $\mathbf{k}$. A flat family $(\mathcal{X}, \mathcal{D})/T \xrightarrow{g} W$ of stable maps over T is given by flat $\mathbb{Q}$-Gorenstein deformation family $(\mathcal{X}, \mathcal{D}) \to T$ of s.l.c. log surface pairs such that g is stable map on each fiber $t \in T$; and there exists an $N > 0$ such that the sheaf $\mathcal{L}_N := \mathcal{O}(N(K_{\mathcal{X}/T} + \mathcal{D} + 5H))$ is a line bundle.

Fixing $K^2 \in \mathbb{Q}$, and $I \subset [0,1]$ a finite set of coefficients of D closed under addition, and $0 < A, B \in \mathbb{Q}$. We define the moduli stack $M := \overline{M}_{K^2,A,B}(W)$ to be the moduli functor from the category $\mathrm{Sch}_{\mathbf{k}}$ of schemes over $\mathbf{k}$ to groupoids, such that $M(T)$ is the $\mathbb{Q}$-Gorenstein deformation families $\{(\mathcal{X},\mathcal{D})/T \to W\}$ of stable maps from s.l.c. log surface pairs to W which satisfy the conditions: for any $t \in T$, and any stable map $(\mathcal{X}_t, \mathcal{D}_t \xrightarrow{g} W)$, $(K_{\mathcal{X}_t+\mathcal{D}_t})^2 = K^2$, $(K_{\mathcal{X}_t+\mathcal{D}_t}) \cdot H = A$ and $H^2 = B$. In addition, for sufficient large $r > 0$, the line bundle $T \mapsto \det f_*\left(\mathcal{O}_{\mathcal{X}}(rK_{\mathcal{X}/T} + \mathcal{D} + 5H)\right)$ associated with each family extends to a functorial line bundle on the entire moduli functor, where $f : (\mathcal{X},\mathcal{D}) \to T$ is the family over T.

Theorem 1.1. *(Theorem 2.18) For the fixed invariants K^2, I, A, B above, the moduli functor M is represented by a projective Deligne-Mumford stack M over* $\mathbf{k}$.

Alexeev [3] defined the moduli stack of stable maps to W, where he required that $D = \sum_{i=1}^n D_i$ and all D_i's are reduced. This makes $\mathcal{D} \to T$ flat in the family $(\mathcal{X},\mathcal{D}) \to T$ of s.l.c. log surface pairs. Then Alexeev [3] used the Kollár ampleness lemma [45, §3.9] to prove the projectivity property of the moduli space M. We drop this requirement on the moduli stack M of stable maps, and use the proof in [49, §7], where Kovács-Patakfalvi used the generalized Kollár ampleness lemma proved in [49, Theorem 5.1] to prove the projectivity of the moduli space of log general type varieties. The main reason to use the generalized Kollár ampleness lemma is that even we fix the ambient surface X, the divisor may deform so that the limit divisor in a flat family degenerates and is not flat over T any more (see [49, §2]).

We should point out that if there is a flat family $(\mathcal{X},\mathcal{D}) \to T$ of s.l.c. log surface pairs, and $\mathcal{D} \to T$ is not flat, then at least to the authors' knowledge, there is no good local deformation and obstruction theory for the infinitesimal extension of the family $(\mathcal{X},\mathcal{D}) \to T$. So in our later construction of moduli stack of stable maps from log surface Deligne-Mumford stacks to projective Deligne-Mumford stacks, we require that in the flat family $(\mathcal{X},\mathcal{D}) \to T$, $\mathcal{D} \to T$ is also flat. There are two constructions for the log surface pairs that this condition is satisfied, Kollár-Shepherd-Barron stability with coefficients $a_i \in (\frac{1}{2}, 1] \cap \mathbb{Q}$ in [46, §6.2], and Alexeev stability in [46, §6.4].

1.3. **Moduli space of stable maps from log surface Deligne-Mumford stacks.** Our goal is to construction a virtual fundamental class on the moduli space M of stable maps, but unfortunately this can not be done on the moduli space M. Similar story happened of W is a point in the moduli sapce M, i.e., the moduli space of log general type surfaces, see [38] for the case of moduli space of general type surfaces.

We extend the construction of the moduli stack M by considering the log Deligne-Mumford stack $(\mathfrak{X},\mathfrak{D})$ such that its coarse moduli space (X, D) is a s.l.c. log surface of general type. Examples contain index one covering Deligne-Mumford stacks $(\mathfrak{X},\mathfrak{D})$ in §1.2, and the lci covering Deligne-Mumford stacks $(\mathfrak{X}^{\mathrm{lci}}, D^{\mathrm{lci}})$. An s.l.c. surface X may contain certain simple elliptic and cusp singularities with embedded dimension very high, then the higher obstruction spaces do not vanish [37]. The lci covering Deligne-Mumford stack $\mathfrak{X}^{\mathrm{lci}}$ of an s.l.c. surface X was defined in [38] and only has l.c.i. singularities. Thus the higher obstruction spaces vanish for the lci covering Deligne-Mumford stack $\mathfrak{X}^{\mathrm{lci}}$. The log version of lci covering Deligne-Mumford stacks can be similarly defined.

Let $\mathfrak{W}$ be a projective Deligne-Mumford stack with projective coarse moduli space W. We define the moduli stack of stable maps from log surface Deligne-Mumford stacks $(\mathfrak{X},\mathfrak{D})$ to W. We still fix $K^2 \in \mathbb{Q}$, $I \subset [0,1]$ a finite set, and $0 < A, B \in \mathbb{Q}$. Let

$$M^{\mathrm{tw}} := \overline{M}^{\mathrm{tw}}_{K^2,A,B}(\mathfrak{W}) : \mathrm{Sch}_{\mathbf{k}} \to \text{Groupoids}$$

be the moduli functor that sends T to the isomorphism classes $\{(\mathfrak{X}, \mathfrak{D})/T \xrightarrow{g} \mathfrak{W}\}$ of stable maps from the log surface Deligne-Mumford stacks $(\mathfrak{X}, \mathfrak{D})$ to $\mathfrak{W}$ such that the the map $(\mathcal{X}, \mathcal{D})/T \to W$ on the coarse moduli spaces is stable with the fixed invariants K^2, A, B and coefficients set of $\mathcal{D}$ is I. We have the following result.

Theorem 1.2. *(Theorem 3.7, Theorem 3.10) The moduli functor M^{tw} is represented by a projective Deligne-Mumford stack M^{tw} over $\mathbf{k}$. Moreover, there exists a proper morphism between Deligne-Mumford stacks*

$$f : M^{tw} \to M$$

which induces a proper map on the coarse moduli spaces.

1.4. **Related and future work.** As we mentioned earlier that in the work [11], the authors defined the reduced virtual cycle $[M]^{\mathrm{red}}$ for the moduli space of stable sheaves on a Calabi-Yau 4-fold W. It is therefore interesting to compare the reduced virtual cycle in [11] with the virtual fundamental class constructed in this paper for some Calabi-Yau 4-folds, and the counting surface invariants defined in [11], [26].

Alexeev [5, §7] also asked to construct the quantum cohomology using the counting log surface invariants. In a future work we will consider the moduli space of stable maps from Hacking stable pairs to projective spaces. Rich geometric structures on such moduli spaces may imply better properties for the counting surface invariants, and even the counting curves invariants in Gromov-Witten theory.

1.5. **Convention.** We work over the field of complex numbers $\mathbf{k} = \mathbb{C}$ throughout of the paper. For the notion of algebraic stack and Deligne-Mumford stack, we follow the book [52], [21] and [68]. All Deligne-Mumford stacks are quasi-projective which, from A. Kresch's equivalence condition, means that they can be embedded into a smooth projective Deligne-Mumford stack. Let $D(\mathcal{O}_M)$ be the derived category of coherent modules on the Deligne-Mumford stack M.

We use lci to represent locally complete intersection and l.c.i. for locally complete intersection singularities. Class T-singularities are either rational double point or two dimensional cyclic quotient singularities of the form $\operatorname{Spec} \mathbf{k}[x, y]/\mu_{r^2 s}$, where $\mu_{r^2 s} = \langle \alpha \rangle$ and there exists a primitive $r^2 s$-th root of unity η such that the action is given by: $\alpha(x, y) = (\eta x, \eta^{dsr-1} y)$ and $(d, r) = 1$. When $s = 1$, these are called Wahl singularities.

Recall a normal surface singularity (S, x) is a rational singularity if the exceptional divisor of the minimal resolution is a tree of rational curves. Simple elliptic surface singularities, cusp or degenerate cusp surface singularities were defined in [44, Definition 4.20]. A *simple elliptic singularity* is a normal Gorenstein surface singularity such that the exceptional divisor of the minimal resolution is a smooth elliptic curve. A normal Gorenstein surface singularity is called a *cusp* if the exceptional divisor of the minimal resolution is a cycle of smooth rational curves or a rational nodal curve. A *degenerate cusp* is a non-normal Gorenstein surface singularity S. If $f : X \to S$ is a minimal semi-resolution, then the exceptional divisor is a cycle of smooth rational curves or a rational nodal curve. In this case S has no pinch points and the irreducible components of S have cyclic quotient singularities.

1.6. **Outline.** Here is an outline for this paper. In §2 we review basic notion of semi-log-canonical log surface pairs, and define the moduli space of stable maps from s.l.c. log surface pairs to a smooth projective scheme. In §3 we construct the moduli stack of stable maps from log surface

Deligne-Mumford stacks to a projective Deligne-Mumford stack. We mainly focus on two log surface Deligne-Mumford stacks: the index one cover Deligne-Mumford stack and the lci cover Deligne-Mumford stack. We prove the coarse moduli space is a projective scheme. In §4 and §5 we study two examples of moduli space of stable maps for log surface Deligne-Mumford stacks. One is Hassett's moduli space of log surface pairs of degree 4 curves in $\mathbb{P}^2$; and the other is the moduli space of stable maps from $(\mathbb{P}^2, C_4)$ to a projective space $\mathbb{P}^N$.

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2. Log surface pairs and their moduli spaces

2.1. **Notations.** All surfaces X in this paper are assumed to be projective and demi-normal; i.e., satisfying Serre's condition S_2. Any such a surface X has only normal crossing singularities in codimension 1.

Let X be a projective surface. For any coherent sheaf $\mathcal{F}$ on X, $\mathcal{F}^*$ denotes the dual of $\mathcal{F}$, and $\mathcal{F}^{**}$ is the reflexive hull of $\mathcal{F}$. There is a natural morphism $\mathcal{F} \to \mathcal{F}^{**}$, and $\mathcal{F}$ is called reflexive if $\mathcal{F} \to \mathcal{F}^{**}$ is an isomorphism. For a coherent sheaf $\mathcal{F}$, we set

$$\mathcal{F}^{[r]} := (\mathcal{F}^{\otimes r})^{**}.$$

Let ω_X be the dualizing sheaf of X, and K_X the canonical class, then

$$\omega_X^{[r]} = (\omega_X^{\otimes r})^{**} = \mathcal{O}_X(r \cdot K_X),$$

see [66, Appendix to §1] for reference. The sheaf ω_X is torsion free and of rank one. If X is normal, ω_X is a divisorial sheaf which satisfies the equivalent conditions in [66, Appendix to §1, Proposition 2]. Moreover, ω_X is reflexive if X is normal.

Let $\mathcal{G}$ be another coherent sheaf over X, then we denote

$$\mathcal{F}[\otimes]\mathcal{G} := (\mathcal{F} \otimes \mathcal{G})^{**}; \quad \mathrm{Sym}^{[r]}(\mathcal{F}) := (\mathrm{Sym}^r(\mathcal{F}))^{**}.$$

2.2. **Log surfaces of general type.**

Definition 2.1. *A log surface pair (X, D) consists of a projective surface X, and an effective $\mathbb{Q}$-divisor $D \subset X$. A* simple normal crossing (snc) *surface pair (or a* log smooth *surface pair) is a log surface pair (X, D) such that X is smooth and* $\mathrm{Supp}(D)$ *is a simple normal crossing divisor.*

A stable *log surface pair (X, D) is a log surface pair such that*

(1) X is proper;
(2) (X, D) has s.l.c. singularities, which means that if $\nu : X^\nu \to X$ is the normalization of X, and let Δ^ν, D^ν be the inverse images of the double curve $\Delta \subset X$ and $D \subset X$. Then the pair $(X^\nu, \Delta^\nu + D^\nu)$ has log canonical singularities;
(3) the $\mathbb{Q}$-Cartier divisor $K_X + D$ is ample.

Our reference for this definition is [48, §5.10].

We introduce the index one cover. Let (X, D) be a s.l.c. log surface pair[1] and $x \in X$ a singularity germ. Recall that in [29, §2.3] and [44], the index of the singularity $x \in X$ is the smallest integer r such that $\omega_X^{[r]}$ is invertible[2] around x. In an étale neighborhood of $x \in X$, we perform the following construction. We fix an isomorphism around x,

$$\theta : \omega_X^{[r]} \to \mathcal{O}_X,$$

and define

$$Z := \mathrm{Spec}_{\mathcal{O}_X}(\mathcal{O}_X \oplus \omega_X^{[1]} \oplus \cdots \oplus \omega_X^{[r-1]}) \xrightarrow{\pi} X$$

where the multiplication on $\mathcal{O}_Z$ is given by the isomorphism θ. Then we get a cyclic cover $\pi : Z \to X$ of degree r called the index one cover. Let $y \in Z$ be the inverse image of x, then $\pi : Z \setminus \{y\} \to X \setminus \{x\}$ is an étale cover. The surface Z is Gorenstein and s.l.c. In particular, the dualizing sheaf ω_Z is invertible. Note that if $\omega_X^{[N]}$ is invertible for some $N > 0$, then $r|N$.

The index one cover $Z \to X$ is determined only by the étale topology. Thus, the local data of index one covers everywhere around X determine a Deligne-Mumford stack

$$\pi : \mathfrak{X} \to X$$

with X the coarse moduli space. We call $\mathfrak{X}$ the index one covering Deligne-Mumford stack, which is locally given by $[Z/\mathbb{Z}_r]$.

We define the notion of $\mathbb{Q}$-Gorenstein deformation families $(\mathcal{X}, \mathcal{D})/T$ of s.l.c. log surface pairs over a scheme T of finite type over $\mathbf{k}$.

Definition 2.2. *We say that a deformation $(\mathcal{X}, \mathcal{D})/T$ over a scheme T of s.l.c. log surface pairs is* $\mathbb{Q}$-Gorenstein, *if around a singular germ (X, D, x) the deformation is induced by an $\mathbb{Z}_r$-equivalent deformation of the index one cover Z of $x \in X$. That is, it is induced by a deformation of the index one covering Deligne-Mumford stack $[Z/\mathbb{Z}_r]$.*

Let $(\mathcal{X}, \mathcal{D})/T$ be a flat $\mathbb{Q}$-Gorenstein deformation family of log surface pairs, and $\omega_{\mathcal{X}/T}$ be its relative dualizing sheaf. From [44, §5.4], [66, Appendix to §1], and [29, §3.1], let

$$\omega_{\mathcal{X}/T}^{[N]} := (\omega_{\mathcal{X}/T}^{\otimes N})^{**} = i_*(\omega_{\mathcal{X}^0/T}^{\otimes N})$$

where $i : \mathcal{X}^0 \hookrightarrow \mathcal{X}$ is the inclusion of the Gorenstein locus,; i.e., the locus where the relative dualizing sheaf $\omega_{\mathcal{X}/T}$ is invertible. If r is the index of the singularity $x \in X$, then $r|N$. Suppose that $\mathcal{Z}/(0 \in T)$ is a $\mathbb{Z}_r$-equivariant deformation of Z inducing a $\mathbb{Q}$-Gorenstein deformation $\mathcal{X}/(0 \in T)$ of X, then we have that

$$\mathcal{Z} = \mathrm{Spec}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}} \oplus \omega_{\mathcal{X}/T}^{[1]} \oplus \cdots \oplus \omega_{\mathcal{X}/T}^{[r-1]}),$$

where the multiplication of $\mathcal{O}_{\mathcal{Z}}$ is given by fixing a trivialization of $\omega_{\mathcal{X}/T}^{[r]}$. It is a family of deformations from [29, Lemma 2.5].

The index one cover of the germ $(\mathcal{X}, x)$ is also uniquely determined by the étale topology. Thus, the data of the index one covers everywhere locally on $\mathcal{X}/T$ glue to define a Deligne-Mumford stack $\mathfrak{X}/T$ which we call the index one covering Deligne-Mumford stack over T. Also the relative

[1] I.e. log surface pair satisyfing (1) and (2) in Definition 2.1.

[2] I.e. rK_X is Cartier around x.

dualizing sheaf $\omega_{\mathfrak{X}/T}$ is invertible. If $\mathcal{D} \subset \mathcal{X}$ avoids the generic codimension one singular points of $\mathcal{X}_t$ for each $t \in T$, then let $\mathfrak{D} := \pi^{-1}(\mathcal{D})$, and we get a $\mathbb{Q}$-divisor $\mathfrak{D}$ in $\mathfrak{X}$, where $\pi : \mathfrak{X} \to \mathcal{X}$ is the map to its coarse moduli space.

Remark 2.3. *From the classification of s.l.c. surface singularities in* [44, Theorem 4.23, Theorem 4.24], *there are only finitely many isolated singular points in a s.l.c. surface X whose local indices are bigger than 1. We can take N to be the least common multiple of all the local indices, then $\omega_X^{[N]}$ is invertible.*

If there is a flat $\mathbb{Q}$-Gorenstein family $(\mathcal{X}, \mathcal{D})/T$ of s.l.c. log surface pairs, then there exists a uniform $N > 0$ such that $\omega_{\mathcal{X}/T}^{[N]}$ is invertible. This follows from the boundedness result in [4], [30, Theorem 1.1].

2.3. **Stable maps from log surface pairs and the moduli space.** We define stable maps from a s.l.c. log surface pair (X, D) to a smooth projective variety W with a polarization $\mathcal{O}_W(1)$, which generalizes stable maps definition of Alexeev in [3].

Definition 2.4. *A stable map from a log surface pair (X, D) to W is given by a morphism*

$$g : (X, D) \to W$$

such that

(1) $D = \sum_{i=1}^n a_i D_i$ is a $\mathbb{Q}$-divisor on X;
(2) the pair (X, D) has s.l.c. singularities;
(3) the divisor $K_X + D$ is relatively g-ample;
(4) let $H := g^*\mathcal{O}_W(1)$, then there exists $N > 0$ such that the sheaf $L_N := \mathcal{O}(N(K_X + D + 5H))$ is a line bundle.

Definition 2.5. *Let T be a scheme of finite type over $\mathbf{k}$. A flat family of stable maps from log surface pairs (X, D) to W over T is given by*

$$g : (\mathcal{X}, \mathcal{D})/T \to W$$

such that $(\mathcal{X}, \mathcal{D}) \to T$ is a $\mathbb{Q}$-Gorenstein deformation flat family of s.l.c. log surface pairs and g is a morphism satisfying

(1) $\mathcal{D} \subset \mathcal{X}$ is a $\mathbb{Q}$-divisor such that $\mathcal{D}|_t = D_t \subset \mathcal{X}_t$ is the $\mathbb{Q}$-divisor of the log pair $(\mathcal{X}_t, D_t)$;
(2) the divisor $K_{\mathcal{X}/T} + \mathcal{D}$ is relatively g-ample;
(3) there exists $N > 0$ such that the sheaf $\mathcal{L}_N := \mathcal{O}(N(K_{\mathcal{X}/T} + \mathcal{D} + 5H))$ is a line bundle.

We mainly follow [3] and [49, §6] to define the moduli functor of stable maps. Note that [3] required that for any family of stable maps $(\mathcal{X}, \mathcal{D}) \to T$, the divisor $D = \sum_{i=1}^n D_i$ has components D_i all reduced. This makes the family $\mathcal{D} \to T$ is also flat. In the definition below, we drop this condition but require that the base T is normal.

Definition 2.6. *Fix $K^2 \in \mathbb{Q}_{>0}$, and $I \subset [0, 1]$ a finite set of coefficients closed under addition, and $A, B \in \mathbb{Q}_{>0}$. Define a functor*

$$M := \overline{M}_{K^2,A,B}(W) : \mathrm{Sch}_{\mathbf{k}} \to \textit{Groupoids}$$

whose value on a normal scheme T is given by

(2.3.1)

$$M(T) = \left\{ \begin{array}{c|l} & (1)\ (\mathcal{X},\mathcal{D})/T \xrightarrow{g} W \text{ is a family of stable maps in Definition 2.5;} \\ & (2)\ \mathcal{D} \text{ is a Weil divisor on } \mathcal{X} \text{ which does not meet with the generic and} \\ & \quad\ \text{the codimension one singular points of } \mathcal{X}_t \text{ for } t \in T; \\ (\mathcal{X},\mathcal{D}) \xrightarrow{g} W & (3) \text{ For each geometric point } t \in T, \text{ we have} \\ f \downarrow \quad & \quad\ (\text{coefficients of } \mathcal{D}_t) \subset I \text{ and } (K_{\mathcal{X}_t} + \mathcal{D}_t)^2 = K^2; \\ T \quad & (4) \text{ For each point } t \in T, \text{ we have } (K_{\mathcal{X}_t} + \mathcal{D}_t) \cdot H = A, H^2 = B. \\ & \quad\ \text{Here } H = g^*\mathcal{O}_W(1). \end{array} \right\}$$

Also the line bundle $T \mapsto \det f_*(\mathcal{O}_{\mathcal{X}}(r(K_{\mathcal{X}/T} + \mathcal{D} + 5H)))$ associated with each family extends to a functorial line bundle on the entire functor for sufficiently large divisible integer $r > 0$.

Remark 2.7. *The boundedness of family of stable maps follows from* [3], [30, Theorem 1]. *Thus, there exists a finite set $I_0 \subset I$ containing all possible coefficients of $\mathcal{D}$, and there also exists a uniform $N > 0$ such that $\mathcal{L}_N = \mathcal{O}_{\mathcal{X}}(N(K_{\mathcal{X}/T} + \mathcal{D} + 5H))$ is a line bundle for each family $(\mathcal{X},\mathcal{D})/T \to W$.*

If we only care about the component of the moduli space containing smooth log pairs (X, D), we can introduce a subfunctor.

Definition 2.8. *We define the subfunctor $M^{sm} \subset M$ as the functor M above, plus one additional condition:*

• *for each $s \in T$, there exists a one-dimensional family $\mathcal{X} \to T' \subset T$ from M with $\mathcal{X}_s$ the central fiber, and an irreducible general fiber $\mathcal{T}_t$ such that*

(1) *$\mathcal{X}_t$ is irreducible;*
(2) *the $(\mathcal{X}_t, D_t)$ is Kawamata-log-terminal (k.l.t.).*

Remark 2.9. *When W is a point, Definition 2.6 is* [49, Definition 6.2]. *When W is a projective variety of dimension ≥ 1, our definition generalizes* [3, Definition 3.17].

2.4. **Projectivity of the moduli stack.** We first prove the boundedness for the family of stable maps. This is from the boundedness of families of log surface pairs, and a generalization of [4, Theorem 3.14].

Theorem 2.10. *(*[4, §9.2]*) Let $I \subset [0,1]$ be a set satisfying the descending chain condition. For all the Then the class $\{(X, D)\}$ consisting of s.l.c. surface pairs (X, D) with $\mathbb{Q}$-divisors $D = \sum a_j D_j$ such that*

- *$a_j \in I$,*
- *$K_X + D$ is ample, and*
- *$(K_X + D)^2 = K^2$ is fixed,*

is bounded.

Proposition 2.11. *Let $\{(\mathcal{X},\mathcal{D})/T \xrightarrow{g} W\}$ be the class of families of stable maps to W in Definition 2.6. Then the class is bounded.*

Proof. We apply Theorem 2.10 to the set I, and to the divisor $K_X + D + 4H$, where $4H \in |4H|$ is a general divisor. The pair $(X, D + 4H)$ is also s.l.c. Thus, there exists a large $N > 0$, such that (X, D) can be embedded into a projective space $\mathbb{P}^M$ by the linear system $|N(K_X + D + 4H)|$. Let

$g:(X,D)\to W$ be a stable map, and let $W\to\mathbb{P}^{M_1}$ be the embedding of W by $|\mathcal{O}_W(M_1)|$. Then the graph

$$\Gamma_g\hookrightarrow\mathbb{P}^M\times\mathbb{P}^{M_1}$$

is the Segre embedding. Note that

$$\mathcal{O}_{\mathbb{P}^M\times\mathbb{P}^{M_1}}(1)|_{X\cong\Gamma_g}=L_N=\mathcal{O}(N(K_X+D+5H)).$$

Since L_N^2 is fixed, the boundedness Theorem 2.10 above implies that all the maps $\{(X,D)\stackrel{g}{\to}W\}$ in the class are parametrized by finite many products of Hilbert schemes. Then we fix to the subscheme parametrizing subschemes of $\mathbb{P}^M\times W$ fixing $\mathcal{O}_{\mathbb{P}^M}(1)^2,\mathcal{O}_{\mathbb{P}^M}(1)\cdot\mathcal{O}_{\mathbb{P}^{M_1}}(1),\mathcal{O}_{\mathbb{P}^{M_1}}(1)^2$, and thus is also a finite scheme. □

Our moduli functor M considers the log surface pairs (X,D) such that $D=\sum a_jD_j$ for $a_j\in I$. As we mentioned earlier in the introduction, in this case even we have a flat family $(\mathcal{X},\mathcal{D})\to T$ of s.l.c. log surface pairs over a scheme T, the divisor $\mathcal{D}\to T$ may not be flat. To overcome the difficulty of families of log surface pairs over non-reduced schemes, [49, §6.A] defined another moduli functor, a bit larger than Definition 2.6 which avoids the above difficulty. We generalize it to the case of stable maps.

Definition 2.12. *We fix $h:\mathbb{Z}\to\mathbb{Z}$ a polynomial, and $N\in\mathbb{Z}_{>0}$ depending on h, and define a moduli functor:*

$$\overline{M}_h^N(W):\mathrm{Sch}_{\mathbf{k}}\to Groupoids$$

by: for any normal scheme T, $M(T)=$

(2.4.1)

$$\left\{\begin{array}{c|l} & \text{(1) } f:\mathcal{X}\to \text{ is a flat family, and } g:(\mathcal{X},\mathcal{D})\to W \text{ is a map;}\\ & \text{(2) } \mathscr{L}(5H) \text{ is a relatively very ample line bundle on } \mathcal{X} \text{ such that}\\ & \quad R^if_*(\mathscr{L}(5H))^r=0 \text{ for } r>0;\\ & \text{(3) For each geometric point } t\in T, \text{ we have}\\ \begin{array}{l}\mathcal{X}\xrightarrow{g}W,\ \phi:\omega_{\mathcal{X}/T}^{\otimes N}\to\mathscr{L}\\ {\scriptstyle f}\big\downarrow\\ T\end{array} & \quad \text{a.) } \phi \text{ is an isomorphism at the generic points and at the}\\ & \qquad \text{codimension one singular points of } \mathcal{X}_t, \text{ hence, it determines}\\ & \qquad \text{a divisor } D_t \text{ such that } \mathscr{L}_t\cong\mathcal{O}_{\mathcal{X}_t}(N(K_{\mathcal{X}_t}+D_t)) \text{ and}\\ & \qquad g:(\mathcal{X}_t,D_t)\to W \text{ is a stable map as in Definition 2.5;}\\ & \quad \text{b.) } (\mathcal{X}_t,D_t) \text{ is s.l.c.;}\\ & \quad \text{c.) } h(r)=\chi(\mathcal{X}_t,(\mathscr{L}_t(5H))^r) \text{ for every integer } r>0.\end{array}\right\}$$

where

(1) ϕ corresponds to a divisor $\mathcal{D}$ such that

$$\mathcal{O}_{\mathcal{X}}(N(K_{\mathcal{X}/T}+\mathcal{D}+5H))\cong\mathscr{L}(5H).$$

(2) let $(\mathcal{X}/T\to W,\phi:\omega_{\mathcal{X}/T}^{\otimes N}\to\mathscr{L})$ and $(\mathcal{X}'/T'\to W,\phi':\omega_{\mathcal{X}'/T'}^{\otimes N}\to\mathscr{L}')$ be two objects in the functor, a morphism between them is given by the following Cartesian diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "\alpha"] \arrow[d] & \mathcal{X} \arrow[d] \\ T' \arrow[r] & T \end{tikzcd}$$

and an isomorphism $\xi : \alpha^*\mathscr{L} \to \mathscr{L}'$ which fits the following commutative diagram:

(2.4.2)

$$\begin{tikzcd}
\alpha^*\omega^{\otimes N}_{\mathcal{X}/T} \arrow[r] \arrow[rr, bend left, "\alpha^*\phi"] & (\alpha^*\omega^{\otimes N}_{\mathcal{X}/T})^{**} \arrow[r] \arrow[d, "\cong"'] & \alpha^*(\mathscr{L}) \arrow[d, "\xi"] \\
\omega^{\otimes N}_{\mathcal{X}'/T'} \arrow[r] \arrow[rr, bend right, "\phi'"'] & (\alpha^*\omega^{\otimes N}_{\mathcal{X}'/T'})^{**} \arrow[r] & \mathscr{L}'
\end{tikzcd}$$

where above $\cong$ is the unique extension of the canonical isomorphism on the relative Gorenstein locus. Moreover, a morphism above is an isomorphism if $T \to T'$ is identity and α is an isomorphism.

(3) *The following pullback property is fixed. Let $(\mathcal{X}/T \to W, \phi : \omega^{\otimes N}_{\mathcal{X}/T} \to \mathscr{L}) \in \overline{M}^N_h(T)$, and let $T' \to T$ be a* **k***-morphism. Then we have $(\mathcal{X}_{T'}/T' \to W, \phi_{[T']} : \omega^{\otimes N}_{\mathcal{X}_{T'}/T'} \to \mathscr{L}_{T'})$, where $\phi_{[T']}$ is defined via the following commutative diagram:*

$$\begin{tikzcd}
\omega^{\otimes N}_{\mathcal{X}_{T'}/T'} \arrow[d] \arrow[dr, "\phi_{[T']}"] & \\
\omega^{[N]}_{\mathcal{X}_{T'}/T'} \arrow[r] & \mathscr{L}_{T'}. \\
(\omega^{\otimes N}_{\mathcal{X}/T})_{T'} \arrow[u] \arrow[ur, "\phi_{T'}"'] &
\end{tikzcd}$$

This means that via the natural identification, $Hom((\omega^{\otimes N}_{\mathcal{X}/T})_{T'}, \mathscr{L}_{T'}) = Hom(\omega^{\otimes N}_{\mathcal{X}_{T'}/T'}, \mathscr{L}_{T'})$, and $\phi_{T'}$ corresponds to $\phi_{[T']}$.

Let $(\mathcal{X}, \mathcal{D})/T \xrightarrow{g} W$ be a family of stable maps from s.l.c. log surface pairs $\mathcal{X}, \mathcal{D}$ to W. For any element (X, D) there is a stable map, and $K_X + D$ is g-relatively ample. From [3, Lemma 2.23], $K_X + D + 4H$ is ample. Thus, we have a stable log surface pair $(X, D + 4H)$. Therefore $\operatorname{Aut}(X, D + 4H)$ is finite. Basic analysis of stacks in [52], [68] proves that this functor $\overline{M}^N_h(W)$ is an étale stack. Next we show $\overline{M}^N_h(W)$ is a Deligne-Mumford stack, and is projective when N is sufficiently large and divisible.

We first have the following result.

Proposition 2.13. *The functor $\overline{M}^N_h(W)$ is a Deligne-Mumford stack of finite type over* **k**.

Proof. The proof is similar to [49, Proposition 6.11], following the definition of Deligne-Mumford stacks in [21, 4.21].

We need to prove that the diagonal morphism

$$\overline{M}^N_h(W) \to \overline{M}^N_h(W) \times_{\mathbf{k}} \overline{M}^N_h(W)$$

is representable and unramified, and there exists a smooth étale surjection

$$\mathscr{C} \to \overline{M}_h^N(W)$$

from a scheme of finite type over $\mathbf{k}$.

We first consider the statement on the diagonal. Let T be a scheme, then for any $s, t \in \overline{M}_h^N(W)(T)$ which corresponds to a morphism $T \to \overline{M}_h^N(W) \times_{\mathbf{k}} \overline{M}_h^N(W)$, the fiber product

$$\begin{tikzcd} T \times_{\overline{M}_h^N(W) \times_{\mathbf{k}} \overline{M}_h^N(W)} \overline{M}_h^N(W) \arrow[r] \arrow[d] & \overline{M}_h^N(W) \arrow[d, "\Delta"] \\ T \arrow[r] & \overline{M}_h^N(W) \times_{\mathbf{k}} \overline{M}_h^N(W) \end{tikzcd}$$

is given by the functor $\mathbf{Isom}_T(s,t)$. Thus, let

$$(\mathcal{X}, \phi) := \left(\mathcal{X}/T \xrightarrow{g} W, \phi : \omega_{\mathcal{X}/T}^{\otimes N} \to \mathscr{L}\right)$$

and

$$(\mathcal{X}', \phi') := \left(\mathcal{X}'/T \xrightarrow{g'} W, \phi' : \omega_{\mathcal{X}'/T}^{\otimes N} \to \mathscr{L}'\right)$$

be the two families of stable maps in $\overline{M}_h^N(W)$ corresponding to s, t. We need to prove that $\mathbf{Isom}_T((\mathcal{X}, \phi), (\mathcal{X}', \phi'))$ is represented by a quasi-projective group scheme $\mathrm{Isom}_T((\mathcal{X}, \phi), (\mathcal{X}', \phi'))$ over T. In the family of stable maps, we only require the line bundle $\mathscr{L}$ to satisfy that $\mathscr{L}(5H)$ is relatively very ample. $\mathscr{L}$ itself is not required to be relatively very ample. This small difference does not affect adopting the proof from [49, Lemma 6.8].

Now let S be a scheme, then $\mathbf{Isom}_T((\mathcal{X}, \phi), (\mathcal{X}', \phi'))(S)$ is the set of S-isomorphisms between $(\mathcal{X}, \phi)_S$ and $(\mathcal{X}', \phi')_S$. First the connected components of the Isom scheme $\mathrm{Isom}_T(\mathcal{X}, \mathcal{X}')$ parametrizing isomorphisms $\gamma : \mathcal{X}_S \to \mathcal{X}'_S$ such that $\gamma^* \mathscr{L}'_S \cong \mathscr{L}_S$ is given by $I := \mathrm{Isom}_T^*(\mathcal{X}, \mathcal{X}') \to T$ by [43, Exercise 1.10.2]. We have a universal isomorphism $\alpha : \mathcal{X}_I \to \mathcal{X}'_I$. Let

$$J := \mathrm{Isom}_I(\alpha^* \mathscr{L}'_I, \mathscr{L}_I)$$

be the open subset of $\mathrm{Hom}_I(\alpha^* \mathscr{L}'_I, \mathscr{L}_I)$ parametrizing isomorphisms; see [47, 33]. The space J is also equipped with a universal isomorphism $\xi : \alpha_J^* \mathscr{L}'_J \to \mathscr{L}_J$. The triple

$$(J, \alpha_J : \mathcal{X}_J \to \mathcal{X}'_J, \xi : \alpha_J^* \mathscr{L}'_J \to \mathscr{L}_J)$$

is a fine moduli space which represents the functor

$$S \mapsto \{(\beta, \zeta) | \beta : \mathcal{X}_S \to \mathcal{X}'_S \text{ and } \zeta : \beta^* \mathscr{L}'_S \xrightarrow{\cong} \mathscr{L}_S \text{ an isomorphism}\}.$$

The functor $\mathbf{Isom}_T((\mathcal{X}, \phi), (\mathcal{X}', \phi'))$ is the functor J, plus the following commutative diagram

$$\begin{tikzcd} \beta^* \omega_{\mathcal{X}'_S/S}^{\otimes N} \arrow[r, "\beta^* \phi'_{[S]}"] \arrow[d, "\cong"'] & \beta^* \mathscr{L}'_S \arrow[d, "\zeta"] \\ \omega_{\mathcal{X}_S/S}^{\otimes N} \arrow[r, "\phi_{[S]}"] & \mathscr{L}_S. \end{tikzcd} \tag{2.4.3}$$

The same proof as in [49, Proposition 6.8] shows that the extra condition (2.4.3) is a closed condition. Therefore, the functor $\mathbf{Isom}_T((\mathcal{X},\phi),(\mathcal{X}',\phi'))$ is represented by a quasi-projective group scheme $\mathrm{Isom}_T((\mathcal{X},\phi),(\mathcal{X}',\phi'))$. It is unramified over $\mathbf{k}$ since its geometric fibers are finite, which is due to the fact that a stable map $(X,D)\xrightarrow{g}W$ has a finite automorphism group.

Next we show that there is a smooth surjection

$$\mathscr{C}\to\overline{M}_h^N(W)$$

from a scheme $\mathscr{C}$ of finite type over $\mathbf{k}$. First there exists some $M>0$ depending on $h(r)$, such that every pair $(X,\phi:\omega_X^{\otimes N}\to\mathscr{L})$ occured in the Hilbert scheme $\mathrm{Hilb}^h_{\mathbb{P}^M}$, so that $X\hookrightarrow\mathbb{P}^M$ is given by $H^0(X,\mathscr{L})$. For any stable map $(X,D)\xrightarrow{g}W$, its graph Γ_g satisfies $\Gamma_g\subset\mathbb{P}^M\times\mathbb{P}^{M_1}$ for some $M_1>0$; see Proposition 2.11. Then every $(X\xrightarrow{g}W,\phi:\omega_X^{\otimes N}\to\mathscr{L})\in\overline{M}_h^N(W)$ is contained in $\mathrm{Hilb}^h_{\mathbb{P}^M\times\mathbb{P}^{M_1}}$. Let first $\mathcal{H}_1\subset\mathrm{Hilb}^h_{\mathbb{P}^M\times\mathbb{P}^{M_1}}$ be the open subscheme parametrizing $X\subset\mathbb{P}^M$ such that $H^i(X,\mathcal{O}_X(r))=0$ for $i>0,r>0$. From [49, Proposition 6.11], we let

$$\mathcal{H}_2\subset\mathcal{H}_1$$

be the open subscheme parametrizing the demi-normal surfaces (where demi-normal surfaces are equidimensional and reduced). Let $(\mathcal{M}_2\xrightarrow{g}W)$ be the universal family over $\mathcal{H}_2$. In the scheme $\mathcal{H}_2$, we let $\mathcal{H}_2^0\subset\mathrm{Hilb}^h_{\mathbb{P}^M}$ be the subscheme only corresponding to $(X,D)\subset\mathbb{P}^M$. Then [47, 33] again implies that there exists a fine moduli space

$$M_3:=\mathrm{Hom}_{\mathcal{H}_2}(\omega^{\otimes N}_{\mathcal{M}_2/\mathcal{H}_2},\mathcal{O}_{\mathcal{M}_2}(1)).$$

Let $\mathcal{M}_3\to M_3$ and $\mathcal{O}_{\mathcal{M}_3}(1)$ be the natural pullbacks from $\mathcal{M}_2$ and $\mathcal{O}_{\mathcal{M}_2}(1)$. We have a universal morphism $\gamma:\omega^{\otimes N}_{\mathcal{M}_3/\mathcal{H}_3}\to\mathcal{O}_{\mathcal{M}_3}(1)$. Let $M_4\subseteq M_3$ be the open subscheme where γ is an isomorphism at every generic point and singular codimension one point. Still let $\mathcal{M}_4$ and $\mathcal{O}_{\mathcal{M}_4}(1)$ be the restriction of $\mathcal{M}_3$ and $\mathcal{O}_{\mathcal{M}_3}(1)$ on M_4. Finally let

$$M_5:=\{t\in M_4\mid\omega^{\otimes N}_{(\mathcal{M}_4)_t}\to\mathcal{O}_{(\mathcal{M}_4)_t}(1)\text{ corresponds to a s.l.c. pair map}\}.$$

Let $\mathscr{C}:=M_5$ and let

$$\left(\widetilde{\mathscr{C}}/\mathscr{C}\xrightarrow{g}W,\phi:\omega^{\otimes N}_{\widetilde{\mathscr{C}}/\mathscr{C}}\to\mathcal{O}_{\widetilde{\mathscr{C}}}(1)\right)$$

be the restriction of $\mathcal{M}_4\to M_4$ and γ over M_5. Then this scheme $\mathscr{C}$ is of finite type, and every family of stable maps in $\overline{M}_h^N(W)$ corresponds to a family in M_5. There is a smooth morphism:

$$\mathscr{C}\to\overline{M}_h^N(W)$$

induced by the universal family. Its smoothness follows by showing that it is formally smooth, as in the proof of [49, Proposition 6.11]. ☐

Proposition 2.14 (completeness)**.** *The moduli functor M is* complete*. Namely, let T be the spectrum of a DVR with general point T_{gen} and $((\mathcal{X}_{\mathrm{gen}},\mathcal{D}_{\mathrm{gen}})/T_{\mathrm{gen}}\to W)\in M(T_{\mathrm{gen}})$, then there is a finite cover $\varphi:T'\to T$, and a stable map $((\mathcal{X}',\mathcal{D}')/T'\to W)\in M(T')$ such that $((\mathcal{X}',\mathcal{D}')/T'\to W)|_{T_{\mathrm{gen}}}=\varphi^*((\mathcal{X}_{\mathrm{gen}},\mathcal{D}_{\mathrm{gen}})/T_{\mathrm{gen}}\to W)$.*

Proof. This follows from the valuative criterion for properness, see [45, 2.1.8.]. If there is a flat family of stable maps

$$((\mathcal{X},\mathcal{D})/T \xrightarrow{g} W) \in M(T),$$

then from V. Alexeev, $K_{\mathcal{X}/T}+\mathcal{D}+4H$ is ample, where $H=g^*\mathcal{O}(1)$. Thus we have a family $((\mathcal{X},\mathcal{D}+4H)/T$ of log surface pairs, which satisfy the valuative criterion for properness, see [44], [45]. Thus, in turn the family $((\mathcal{X},\mathcal{D})/T \xrightarrow{g} W) \in M(T)$ satisfies the valuative criterion for properness. □

2.5. **Proof of Theorem 1.1.** We prove the projectivity property of the moduli stack $M = \overline{M}_{K^2,I,A,B}(W)$. Our moduli space $M \subseteq \overline{M}_h^N(W)$ when N is sufficiently large and divisible. We will show that $\overline{M}_h^N(W)$ is projective when N is sufficiently large.

We already proved that M is bounded. To show that the moduli space M is proper, we need to show the completeness, i.e., the valuative criterion of properness. This is Proposition 2.14. Thus, to prove the projectivity of the coarse moduli space of M or $\overline{M}_h^N(W)$, the Nakai-Moishezon criterion implies that we need to show that: for any family

$$(\mathcal{X},\mathcal{D})/T \stackrel{g}{\to} W$$

of stable maps, for sufficiently divisible $r>0$,

$$\det f_*\left(\mathcal{O}_{\mathcal{X}}(r(K_{\mathcal{X}/T}+\mathcal{D}+5H))\right)$$

is big.

In case $D=\sum_j D_j$ is reduced, [3, Theorem 4.2] showed the projectivity of $\overline{M}_h^N(W)$ by Kollár's ampleness lemma in [45, 3.9]. In the general situation that $D=\sum_j a_jD_j$ for $a_j \in I$, Kollár's ampleness lemma is not enough. [49, Theorem 5.1] generalized the Kollár's ampleness lemma to almost projective varieties, and proved the projectivity for the moduli space of log general type varieties in [49, §7]. We follow the method in [49, §7] and apply it to the case $\overline{M}_h^N(W)$ when N is large enough. We will mainly point out changes in the proof, and refer the readers to [49, §7] for the necessary arguments.

Let us first fix some notations.

• For a family $(\mathcal{X},\mathcal{D})\to T$ of log surface pairs, we let

$$\mathcal{X}^{(r)} := \underbrace{\mathcal{X}\times_T\times\cdots\times_T\mathcal{X}}_{r}$$

and

$$\mathcal{D}^{(r)} := \underbrace{\mathcal{D}\times_T\times\cdots\times_T\mathcal{D}}_{r}$$

be the fiber products. The fiber power guarantees that the family $\mathcal{X}^{(r)}\to T$ will be varying maximally on some components of the moduli space containing $\mathcal{X}$, see [49, Remark 7.2].

• For the family $(\mathcal{X},\mathcal{D})/T \stackrel{g}{\to} W$ of stable maps, recall that our N in the moduli $\overline{M}_h^N(W)$ means that $\mathcal{O}_{\mathcal{X}}(N(K_{\mathcal{X}/T}+\mathcal{D}+5H))$ is a line bundle. Let

$$\mathscr{L}_d := \mathcal{O}_{\mathcal{X}}(dN(K_{\mathcal{X}/T}+\mathcal{D}+5H)).$$

From [49, Notation 7.6], there exists a dense big open subset $U \subseteq T$ over which all the possible unions of the components of $\mathcal{D}$ are flat over T. Thus we get an almost projective variety $U \subseteq T = \overline{U}$ in [49, Theorem 5.1]. Also let

$$\mathcal{D}_c := \sum_{\mathrm{coeff}_E \mathcal{D} = c} E$$

where the sum runs over all prime divisors, and $\mathcal{D} = \sum_{c \in \mathbb{Q}} \mathcal{D}_c$. There is an open set $V \subseteq U$ over which

(1) $\mathcal{D}_c$ is compatible with base-changes as in [49, Definition 7.5];
(2) the scheme theoretic fiber of $\mathcal{D}_c$ over $v \in V$ is reduced and therefore is equal to its divisorial restrictions.

In summary, choose $N > 0$ we have that

(1) $N(K_{\mathcal{X}/T} + \mathcal{D} + 5H)$ is Cartier;
(2) $\mathscr{L}_d = \mathcal{O}_{\mathcal{X}}(dN(K_{\mathcal{X}/T} + \mathcal{D} + 5H))$ is f-very ample, where $f : (\mathcal{X}, \mathcal{D}) \to T$ is the family;
(3) $R^j f_* \mathscr{L}_d = 0, j > 0$;
(4) $(R^j (f|_{\mathcal{D}_j})_* \mathscr{L}_d|_{\mathcal{D}_j})|_V = 0$; and
(5) $(f_* \mathscr{L}_1)|_V \to (f|_{\mathcal{D}_j})_* \mathscr{L}_d|_{\mathcal{D}_j})|_V$ is surjective.

Proposition 2.15. *We use the notations above. If* $\mathrm{Var}(g)$ *is maximal, then for all* $d \gg 0$,

$$\det f_* \mathscr{L}_d \otimes \left(\otimes_{i=1}^n (\det (f|_{\mathcal{D}_i})_* \mathscr{L}_d|_{\mathcal{D}_i}) \right)$$

is big.

Proof. If $(f : (\mathcal{X}, \mathcal{D}) \to T; g : (\mathcal{X}, \mathcal{D}) \to W)$ is a family of stable maps to W, then from [3, Lemma 2.23], $f : (\mathcal{X}, \mathcal{D} + 4H) \to T$ is a family of stable log surface pairs since $(\mathcal{X}, \mathcal{D} + 4H)$ is s.l.c., and $K_{\mathcal{X}/T} + \mathcal{D} + 4H$ is ample. Thus the argument is reduced to the case of [49, Proposition 7.8]. The proof used the generalized ampleness lemma of Kollár to almost projective varieties [49, Theorem 5.1]. Keeping the notations there, the vector bundle is $\mathrm{Sym}^d(f_* \mathscr{L}_1)$, which is a weakly positive vector bundle on T, and the group $G = GL(M+1, \mathbf{k})$, where $M = h^0(\mathscr{L}_1|_{\mathcal{X}_t}) - 1$. Then applying [49, Theorem 5.1] to the open subset $V \subseteq U \subseteq T$ in the notations above, we get the result. □

Using Proposition 2.15, the same proof as in [49, Theorem 7.1] yields:

Theorem 2.16. *Let* $(\mathcal{X}, \mathcal{D})/T \xrightarrow{g} W$ *be a family of stable maps to* W *with maximal variation over a smooth projective variety* T*, then there exists a* $r > 0$*, such that* $K_{\mathcal{X}^{(r)}/T} + \mathcal{D}_{\mathcal{X}^{(r)}} + 5H$ *is big on at least one component of* $\mathcal{X}^{(r)}$*; i.e.,* $\left(K_{\mathcal{X}^{(r)}/T} + \mathcal{D}_{\mathcal{X}^{(r)}} + 5H \right)^{\dim \mathcal{X}^{(r)}} > 0.$

Thus, we have that

Theorem 2.17. *Let* $(\mathcal{X}, \mathcal{D}) \xrightarrow{f} T; (\mathcal{X}, \mathcal{D}) \xrightarrow{g} W$ *be a family of stable maps to* W *with maximal variation over a normal projective variety* T*, then there exists an integer* $q > 0$ *and a proper closed subvariety* $S \subseteq T$*, such that for every integer* $a > 0$*, and closed irreducible subvariety* $Y \subseteq T$ *not contained in* S*, we have*

$$\det f_* \mathcal{O}_{\mathcal{X}}(aq(K_{\mathcal{X}/T} + \mathcal{D} + 5H))|_{\widetilde{Y}}$$

is big, where $\widetilde{Y}$ *is the normalization of* Y.

Proof. Still one can reduce the family of stable maps to the stable family $(\mathcal{X}, \mathcal{D}+4H) \to T$ is log surface pairs. Then [49, Proposition 7.16] goes through. □

Finally we can prove:

Theorem 2.18. *The moduli stack $M = \overline{M}_{K^2,I,A,B}$ is projective over* $\mathbf{k}$.

Proof. Let us consider the coarse moduli space M for the moduli stack. The space M is proper since it is bounded (Proposition 2.11) and complete (Proposition 2.14). Thus, from the Nakai-Moishezon criterion we need to show that for any proper irreducible subspace V of M,

$$c_1\left(\det(f_V)_*\mathcal{O}_{\mathcal{X}_V}(r(K_{\mathcal{X}_V/V} + \mathcal{D}_V + 5H))\right)^{\dim V} > 0, \tag{2.5.1}$$

where f_V is the corresponding family of stable maps over V. This is proven by letting $V \subseteq Z$ for any family

$$\begin{array}{ccc} (\mathcal{X}_Z, \mathcal{D}_Z) & \xrightarrow{g} & W \\ \Big\downarrow f & & \\ Z & & \end{array}$$

of stable maps over a proper normal scheme Z such that each fiber of f is isomorphic to only finitely many others. We prove it by induction on $\dim(Z)$. The zero dimensional case is trivial. Let $\dim(Z) > 0$. By Proposition 2.17, there exists a $q_Z > 0$, and a closed subset $S \subseteq Z$ that does not contain any component of Z, such that for every $a > 0$ and every irreducible closed subset $Y \subseteq Z$ not contained in S,

$$c_1\left(\det f_*\mathcal{O}_{\mathcal{X}_Z}(aq_Z(K_{\mathcal{X}_Z/Z} + \mathcal{D}_Z + 5H))\right)^{\dim Y} > 0.$$

Let $\widetilde{S} \to S$ be the normalization of S. By induction, $\dim S < \dim Z$, then there exists a $q_{\widetilde{S}} > 0$ such that for every $a > 0$, and every every irreducible closed subset $V \subseteq \widetilde{S}$,

$$c_1\left(\det f_*\mathcal{O}_{\mathcal{X}_Z}(aq_{\widetilde{S}}(K_{\mathcal{X}_Z/Z} + \mathcal{D}_Z + 5H))\right)^{\dim V} > 0.$$

Then we take

$$q := \max\{q_Z, q_{\widetilde{S}}\}$$

and we get (2.5.1). □

Remark 2.19. *It is interesting to have examples of stable map spaces. The most basic start is to consider the stable map spaces from $(\mathbb{P}^2, D)$ to projective space $\mathbb{P}^m$, comparing with the work in* [33], [29].

3. Stable maps from log surface Deligne-Mumford stacks

In this section we study the lifting of the stable maps in §2 to the log surface Deligne-Mumford stacks. We define moduli stacks of stable maps from log surface Deligne-Mumford stacks to a projective Deligne-Mumford stack $\mathfrak{W}$.

3.1. Log surface Deligne-Mumford stack.

Definition 3.1. *A log surface Deligne-Mumford stack is a pair $(\mathfrak{X}, \mathfrak{D})$, where $\mathfrak{X}$ is a projective Deligne-Mumford stack of dimension 2, and $\mathfrak{D} \subset \mathfrak{X}$ is a one-dimensional closed substack. The coarse moduli space (X, D) of $(\mathfrak{X}, \mathfrak{D})$ is a log surface pair. A s.l.c. log surface Deligne-Mumford stack is a log surface Deligne-Mumford stack $(\mathfrak{X}, \mathfrak{D})$ such that (X, D) is an s.l.c. surface pair.*

3.1.1. *The index one covering log pair.* Let (X, D) be a log surface pair. From §2.2, we have the index one covering Deligne-Mumford stack $\pi : \mathfrak{X} \to X$. Let $\mathfrak{D} = \pi^{-1}(D)$, then $(\mathfrak{X}, \mathfrak{D})$ is a log surface Deligne-Mumford stack pair.

3.1.2. *The* lci *covering log pair.* Let (X, D) be a s.l.c. log general type surface pair. From [46, §2.20], the classification of s.l.c. log surface singularities is the same as the classification of s.l.c. surface singularities in [44, Theorem 4.23, Theorem 4.24]. The difference is that the divisor D does not intersect with some bad isolated singularities. Let us write them down as follows:

(1) the semi-log-terminal (s.l.t.) singularities;
(2) the Gorenstein surface singularities, which are either semi-canonical (which are smooth, normal crossing or pinch points or Du Val singularities) or has simple elliptic singularities, cusp, or degenerate cusp singularities;
(3) the $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$-quotients of simple elliptic singularities;
(4) the $\mathbb{Z}_2$-quotients of cusps and degenerate cusp singularities.

The s.l.t. surface singularities are quotient singularities or normal crossing singularities. From [44, Proposition 3.10], if a quotient singularity germ (X, x) admits a $\mathbb{Q}$-Gorenstein deformation, then this quotient singularity must be a class T-singularity. Therefore, its index one cover has A_n-type singularities, hence is l.c.i. Thus, from the above classification, [38, Proposition 4.14], and [46, §2.20] that the divisor D does not intersect with any simple elliptic singularities, cusp, or degenerate cusp singularities.

We briefly review the construction of lci covers in [38, §5.3]. Suppose that (X, D, x) is a s.l.c. log surface singularity germ such that $x \in X$ is: case 1: a simple elliptic singularity, a cusp or a degenerate cusp singularity; or case 2: the $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$-quotients of a simple elliptic singularity, and the $\mathbb{Z}_2$-quotients of a cusp, or a degenerate cusp singularity. Both these two cases are called log canonical surface singularities, and $x \notin D$.

Case 1: In this case the singularity germ (X, D, x) has index one. In [38, §5.3.2], a lci cover

$$(\widetilde{X}, x) \to (X, x)$$

was constructed with transformation group G_x using the theory of Neumann-Wahl [58]. The finite group G_x may be non-abelian. The cover is determined by the link Σ of the singularity germ, where $\Sigma = \partial U$ for the neighborhood $x \in U \subseteq X$. The link Σ is an oriented 3-manifold over $\mathbb{R}$. In the case that x is a cusp singularity, the link Σ is not a rational homology sphere, and is a T^2-bundle over the circle S^1 and $H_1(\Sigma, \mathbb{Z}) = \mathbb{Z} \oplus H_1(\Sigma, \mathbb{Z})_{\text{tor}}$. In the case that x is a simple elliptic singularity, the link Σ is a Seifert fibered 3-manifold (i.e., a circle bundle over the torus T^2). We have in this case $H_1(\Sigma, \mathbb{Z}) = \mathbb{Z}^2 \oplus H_1(\Sigma, \mathbb{Z})_{\text{tor}}$. We don't give the details here, and refer the readers to [38, §5]. The lci cover $(\widetilde{X}, x)$ has only l.c.i. singularities, and in general is an analytic cover.

Case 2: In this case the singularity germ (X, x) has index > 1, and is a rational singularity. In the case of the $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$-quotients of a simple elliptic singularity, the index of the singularity is $2, 3, 4, 6$ respectively, and the index one cover is the simple elliptic singularity; and in the case of the $\mathbb{Z}_2$-quotients of a cusp, or a degenerate cusp singularity, the index is 2 and the index one cover is the cusp singularity and degenerate cusp singularity. The link Σ of the singularity is a rational homology sphere. From [38, §5.3.1], there is a universal abelian cover

$$(\widetilde{X}, x) \to (X, x)$$

with transformation group $G_x = H_1(\Sigma, \mathbb{Z})$, which is a finite abelian group. If $(X, x) = (\overline{X}, x)/\mathbb{Z}_r, (r = 2, 3, 4, 6)$ is the quotient singularity, where $(\overline{X}, x)$ is the simple elliptic singularity, the cusp or the degenerate cusp singularity. Then the lci cover $(\widetilde{X}, x) \to (X, x)$ factors through the map $(\overline{X}, x) \to (X, x)$. The lci cover $(\widetilde{X}, x)$ also has l.c.i. singularities.

In both of these two cases, the local data of lci covers are determined by the local analytic topology or étale topology of (X, D). Thus, the local data of lci covers (giving the Deligne-Mumford stack $[\widetilde{X}/G_x]$) glue to form a Deligne-Mumford stack

$$\pi^{\mathrm{lci}} : \mathfrak{X}^{\mathrm{lci}} \to X$$

which we call the lci covering Deligne-Mumford stack. The morphism $\pi^{\mathrm{lci}} : \mathfrak{X}^{\mathrm{lci}} \to X$ factors through the index one cover and we make the following commutative diagram:

$$\begin{tikzcd} \mathfrak{X}^{\mathrm{lci}} \arrow[r, "\phi"] \arrow[dr, "\pi^{\mathrm{lci}}"'] & \mathfrak{X} \arrow[d, "\pi"] \\ & X. \end{tikzcd}$$

We let $\mathfrak{D}^{\mathrm{lci}} := (\pi^{\mathrm{lci}})^{-1}(D)$. Since D does not meet the above bad isolated singularities where we take the lci covers, $\mathfrak{D}^{\mathrm{lci}} \subseteq \mathfrak{X}^{\mathrm{lci}}$ is isomorphic to the divisor $\mathfrak{D} \subseteq \mathfrak{X}$ in the index one covering Deligne-Mumford stack.

Thus we get a log surface Deligne-Mumford stack pair $(\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}})$ whose coarse moduli space is (X, D).

3.2. **Family of log surface Deligne-Mumford stacks.** Let (X, D) be a s.l.c. log surface pair, and $(\mathfrak{X}, \mathfrak{D}) \to (X, D)$ a log surface Deligne-Mumford stack. Let T be a scheme and we are interested in the flat family $(\mathfrak{X}, \mathfrak{D}) \to T$ of log surface Deligne-Mumford stacks over T.

For the log index one covering Deligne-Mumford stack, the flat family can be constructed as follows. Let $(\mathcal{X}, \mathcal{D}) \to T$ be a $\mathbb{Q}$-Gorenstein deformation family of s.l.c. log surface pairs. Recall that around a singularity germ $(x \in \mathcal{X}/T)$, the index one cover is given by

$$\pi : \mathcal{Z} \to \mathcal{X}$$

and $\mathcal{Z} = \mathrm{Spec}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}} \oplus \omega^{[1]}_{\mathcal{X}/T} \oplus \cdots \oplus \omega^{[r-1]}_{\mathcal{X}/T})$ where r is the local index of $x \in \mathcal{X}/T$. Let $\mathcal{D}_{\mathcal{Z}} \subset \mathcal{Z}$ be given by $\pi^{-1}(\mathcal{D}) = \mathcal{D}_{\mathcal{Z}}$. Then $\mathcal{D}_{\mathcal{Z}} \hookrightarrow \mathcal{Z}$ is uniquely determined by $\mathcal{D} \hookrightarrow \mathcal{X}$. This is because the ideal sheaf $I_{\mathcal{D}_{\mathcal{Z}}}$ in $\mathcal{O}_{\mathcal{Z}}$ is S_2 over T and coincides with the pullback of the ideal sheaf $I_{\mathcal{D}}$ in $\mathcal{O}_{\mathcal{X}}$ over the locus where π is étale. The stack structure around x is given by $[\mathcal{Z}/\mu_r]$. Thus, $\mathcal{D} \hookrightarrow \mathcal{X}$ defines a codimension-one closed substack of the index one covering Deligne-Mumford stack $\mathfrak{X}$,

$$\mathfrak{D} \hookrightarrow \mathfrak{X},$$

and we have the flat family $(\mathfrak{X}, \mathfrak{D}) \to T$ of log surface Deligne-Mumford stacks.

Let $T = \mathrm{Spec}(A)$ where A is a $\mathbf{k}$-algebra. Let $\mathcal{N}$ be a finite A-module. Let $(\mathcal{X}, \mathcal{D})/A$ be a $\mathbb{Q}$-Gorenstein deformation family of log surface pairs over $T = \mathrm{Spec}(A)$. We write

$$\iota : \mathcal{D} \hookrightarrow \mathcal{X}; \quad \iota : \mathfrak{D} \hookrightarrow \mathfrak{X}$$

for the inclusions of the divisors. The log surface Deligne-Mumford stack $(\mathfrak{X},\mathfrak{D})$ represents either the log index one covering Deligne-Mumford stack in §3.1.1 or the log lci covering Deligne-Mumford stack $(\mathfrak{X}^{\mathrm{lci}},\mathfrak{D}^{\mathrm{lci}})$ in §3.1.2 if there is no further specification.

From [46, ??], for a log index one covering Deligne-Mumford stack $(\mathfrak{X},\mathfrak{D})$, $\mathcal{D}\hookrightarrow\mathcal{X}$ does not meet any bad s.l.c. singularities, such as simple elliptic, cusp and degenerate cusp singularities, or their cyclic quotients. From the analysis of s.l.c. singularities in [38], the index one covering Deligne-Mumford stack $\mathfrak{X}$, except the simple elliptic, cusp and degenerate cusp singularities, has only locally complete intersection singularities. For a log lci covering Deligne-Mumford stack $(\mathfrak{X}^{\mathrm{lci}},\mathfrak{D}^{\mathrm{lci}})$, the lci covers only take on simple elliptic, cusp and degenerate cusp singularities. Thus the divisor $\mathfrak{D}\hookrightarrow\mathfrak{X}$ is l.c.i.

For any flat family $(\mathfrak{X},\mathfrak{D})/T$ of log surface Deligne-Mumford stack, we define

$$T^i_{\mathrm{QG}}((\mathcal{X},\mathcal{D})/A,\mathcal{N}):=\mathrm{Ext}^i_{\mathfrak{D}}(\iota^*\mathbb{L}^\bullet_{\mathfrak{X}}\to\mathbb{L}^\bullet_{\mathfrak{D}},\mathcal{O}_{\mathfrak{D}}\otimes_A\mathcal{N}). \tag{3.2.1}$$

Let

$$\mathcal{T}^i_{\mathrm{QG}}((\mathcal{X},\mathcal{D})/A,\mathcal{N}):=\mathcal{E}xt^i_{\mathfrak{D}}(\iota^*\mathbb{L}^\bullet_{\mathfrak{X}}\to\mathbb{L}^\bullet_{\mathfrak{D}},\mathcal{O}_{\mathfrak{D}}\otimes_A\mathcal{N})$$

be the corresponding $\mathcal{E}xt$ sheaf. Thus, $T^i_{\mathrm{QG}}((\mathcal{X},\mathcal{D}),\mathcal{O}_{\mathfrak{D}})=T^i_{(\mathfrak{X},\mathfrak{D})}$ is the i-th tangent space of the map $\iota:\mathfrak{D}\hookrightarrow\mathfrak{X}$ defined in [65] and the introduction.

3.2.1. *Deformation and obstruction.* Let $(\mathfrak{X},\mathfrak{D})/A$ be a family of log surface Deligne-Mumford stacks. We have the following results which are similar to [29, Proposition 3.7, Proposition 3.9], where the author proved the case for index one covering Deligne-Mumford stacks.

Proposition 3.2. *Let $(\mathcal{X},\mathcal{D})/A$ be a $\mathbb{Q}$-Gorenstein deformation family of log surface pairs, and $(\mathfrak{X},\mathfrak{D})/A$ the corresponding log surface Deligne-Mumford stack. Suppose that $\overline{A}\to A$ is an infinitesimal extension of A. Let $(\overline{\mathcal{X}},\overline{\mathcal{D}})/\overline{A}$ be a $\mathbb{Q}$-Gorenstein deformation of $(\mathcal{X},\mathcal{D})/A$, and let $(\overline{\mathfrak{X}},\overline{\mathfrak{D}})$ be the corresponding log surface Deligne-Mumford stack. Then*

$$(\overline{\mathcal{X}},\overline{\mathcal{D}})/\overline{A}\mapsto(\overline{\mathfrak{X}},\overline{\mathfrak{D}})$$

gives a bijection between the set of isomorphism classes of $\mathbb{Q}$-Gorenstein deformations of $(\mathfrak{X},\mathfrak{D})/A$ over $\overline{A}$ and the set of isomorphism classes of deformations of the stack $(\mathfrak{X},\mathfrak{D})/A$ over $\overline{A}$.

Proof. First let $(\overline{\mathcal{X}},\overline{\mathcal{D}})/\overline{A}$ be a $\mathbb{Q}$-Gorenstein deformation of $(\mathcal{X},\mathcal{D})/A$, then the log surface Deligne-Mumford stack $(\overline{\mathfrak{X}},\overline{\mathfrak{D}})/\overline{A}$ is a deformation of $(\mathfrak{X},\mathfrak{D})/A$.

Conversely, if $(\overline{\mathfrak{X}},\overline{\mathfrak{D}})/\overline{A}$ is a deformation of $(\mathfrak{X},\mathfrak{D})/A$, then the coarse moduli space $(\overline{\mathcal{X}},\overline{\mathcal{D}})/\overline{A}$ is a $\mathbb{Q}$-Gorenstein deformation of $(\mathcal{X},\mathcal{D})/A$. Also if $(\overline{\mathfrak{X}},\overline{\mathfrak{D}})/\overline{A}$ is a deformation of $(\mathfrak{X},\mathfrak{D})/A$ with coarse moduli space $(\overline{\mathcal{X}},\overline{\mathcal{D}})/\overline{A}$, then the log surface Deligne-Mumford stack $(\overline{\mathfrak{X}}',\overline{\mathfrak{D}}')/\overline{A}$ of $(\overline{\mathcal{X}},\overline{\mathcal{D}})/\overline{A}$ is isomorphic to $(\overline{\mathfrak{X}},\overline{\mathfrak{D}})/\overline{A}$. This is from [29, Proposition 3.7], since $\overline{\mathfrak{X}}/\overline{A}$ is a deformation of $\mathfrak{X}/A$, then $\overline{\mathfrak{X}}'\cong\overline{\mathfrak{X}}$. Since $(\overline{\mathfrak{X}}',\overline{\mathfrak{D}}')$ and $(\overline{\mathfrak{X}},\overline{\mathfrak{D}})$ have the same coarse moduli spaces, then $\overline{\mathfrak{D}}'\cong\overline{\mathfrak{D}}$. ☐

Let A_0 be a $\mathbf{k}$-algebra, and $\mathcal{N}$ a finite A_0-module. Let $A_0+\mathcal{N}$ be the ring $A_0[\mathcal{N}]$ with $\mathcal{N}^2=0$.

Theorem 3.3. *Let $(\mathcal{X}_0,\mathcal{D}_0)/A_0$ be a $\mathbb{Q}$-Gorenstein deformation family of log surface pairs, and $\mathcal{N}$ a finite A_0-module. Then we have*

1.) the set of isomorphism classes of $\mathbb{Q}$-Gorenstein deformations of $(\mathcal{X}_0,\mathcal{D}_0)/A_0$ over $A_0+\mathcal{N}$ is naturally an A_0-module, and is canonically isomorphic to $T^1_{\mathrm{QG}}((\mathcal{X}_0,\mathcal{D}_0)/A_0,\mathcal{N})$.

2.) let $A \to A_0$ be an infinitesimal extension, and let $\overline{A} \to A$ be a further extension with kernel $\mathcal{N}$, an A_0-module. Let $(\mathcal{X}, \mathcal{D})/A$ be a $\mathbb{Q}$-Gorenstein deformation of $(\mathcal{X}_0, \mathcal{D}_0)/A_0$. Then we have that

(1) there exists a canonical element $o((\mathcal{X}, \mathcal{D})/A, \overline{A}) \in T^2_{\mathrm{QG}}((\mathcal{X}_0, \mathcal{D}_0)/A_0, \mathcal{N})$ which vanishes if and only if there exists a $\mathbb{Q}$-Gorenstein deformation $(\overline{\mathfrak{X}}, \overline{\mathfrak{D}})/\overline{A}$ of $(\mathcal{X}, \mathcal{D})/A$ over $\overline{A}$.
(2) if $o((\mathcal{X}, \mathcal{D})/A, \overline{A}) = 0$, then the set of isomorphism classes of $\mathbb{Q}$-Gorenstein deformations $(\overline{\mathfrak{X}}, \overline{\mathfrak{D}})/\overline{A}$ is an affine space under $T^1_{\mathrm{QG}}((\mathcal{X}_0, \mathcal{D}_0)/A_0, \mathcal{N})$.

Proof. This is from [36, Theorem 1.5.1, Theorem 1.7]. □

Theorem 3.4. *Let $(\mathcal{X}_0, \mathcal{D}_0)/A_0 \xrightarrow{g} W$ be a $\mathbb{Q}$-Gorenstein deformation family of s.l.c. log surface pairs, and let $(\mathfrak{X}_0, \mathfrak{D}_0)/A_0 \xrightarrow{g} \mathfrak{W}$ be one lifting of the flat family of stable maps from index one covering log Deligne-Mumford stacks to $\mathfrak{W}$. Let $\mathcal{N}$ be a finite A_0-module. Then we have that*

1.) the set of isomorphism classes of deformations of $(\mathfrak{X}_0, \mathfrak{D}_0)/A_0 \xrightarrow{g} \mathfrak{W}$ over $A_0 + \mathcal{N}$ is naturally an A_0-module, and is canonically isomorphic to $T^1_{\mathrm{QG}}((\mathfrak{X}, \mathfrak{D})/A \xrightarrow{g} \mathfrak{W}, \mathcal{O}_{\mathfrak{D}} \otimes_A \mathcal{N})$.

2.) let $A \to A_0$ be an infinitesimal extension, and let $\overline{A} \to A$ be a further extension with kernel $\mathcal{N}$, an A_0-module. Let $(\mathfrak{X}, \mathfrak{D})/A \xrightarrow{g} \mathfrak{W}$ be a deformation of $(\mathfrak{X}_0, \mathfrak{D}_0)/A_0 \xrightarrow{g} \mathfrak{W}$. Then

(1) there exists a canonical element $o((\mathfrak{X}, \mathfrak{D})/A \xrightarrow{g} \mathfrak{W}, \overline{A}) \in T^2_{\mathrm{QG}}((\mathfrak{X}, \mathfrak{D})/A \xrightarrow{g} \mathfrak{W}, \mathcal{O}_{\mathfrak{D}} \otimes_A \mathcal{N})$ which vanishes if and only if there exists a deformation $(\overline{\mathfrak{X}}, \overline{\mathfrak{D}})/\overline{A} \xrightarrow{g} \mathfrak{W}$ of $(\mathfrak{X}, \mathfrak{D})/A \xrightarrow{g} \mathfrak{W}$ over $\overline{A}$.
(2) if $o((\mathfrak{X}, \mathfrak{D})/A \xrightarrow{g} \mathfrak{W}, \overline{A}) = 0$, then the set of isomorphic classes of deformations $(\overline{\mathfrak{X}}, \overline{\mathfrak{D}})/\overline{A} \xrightarrow{g} \mathfrak{W}$ is an affine space under $T^1_{\mathrm{QG}}((\mathfrak{X}, \mathfrak{D})/A \xrightarrow{g} \mathfrak{W}, \mathcal{O}_{\mathfrak{D}} \otimes_A \mathcal{N})$.

Proof. This is still from [36, Theorem 1.5.1, Theorem 1.7]. □

3.3. **Moduli stack of stable maps from index one covers.** Let $\mathfrak{W} \to W$ be a projective Deligne-Mumford stack over $\mathbf{k}$ with coarse moduli space W.

We define the moduli functor

$$M^{\mathrm{ind}} := \overline{M}^{\mathrm{ind}}_{K^2,I,A,B}(\mathfrak{W}) : \mathrm{Sch}_{\mathbf{k}} \to \mathrm{Groupoids}$$

which sends a scheme $T \in \mathrm{Sch}_{\mathbf{k}}$ to

$$M^{\mathrm{ind}}(T) = \{(\mathfrak{X}, \mathfrak{D})/A \xrightarrow{g} \mathfrak{W}\},$$

the collection of stable maps from the log index one surface Deligne-Mumford stack $(\mathfrak{X}, \mathfrak{D})/T$ to $\mathfrak{W}$ such that the induced maps

$$(\mathcal{X}, \mathcal{D})/A \xrightarrow{g} W$$

on the coarse moduli spaces are stable and satisfy the conditions in Definition 2.6.

Our first result for the moduli functor is:

Theorem 3.5. *The moduli functor $M^{\mathrm{ind}} = \overline{M}^{\mathrm{ind}}_{K^2,I,A,B}(\mathfrak{W})$ is an algebraic stack of finite type over $\mathbf{k}$.*

Proof. We use Artin's criterion in [10] to prove the moduli functor is an algebraic stack. We check the conditions $(1), (2), (3), (4)$ in [10, Theorem 5.3].

(1) Let T_1, T_2, T, T' be spectra of local Artinian rings of finite type over $\mathbf{k}$. Assume that $T \to T_1$ is a closed immersion and that

$$\begin{tikzcd} T \arrow[r] \arrow[d] & T_1 \arrow[d] \\ T_2 \arrow[r] & T' = T_1 \cup_T T_2 \end{tikzcd}$$

is a pushout diagram in $\mathrm{Sch}_{\mathbf{k}}$, then the functor of fiber categories

$$M^{\mathrm{ind}}(T') \xrightarrow{\sim} M^{\mathrm{ind}}(T_1) \times_{M^{\mathrm{ind}}(T)} M^{\mathrm{ind}}(T_2)$$

is an equivalence of groupoids.

The above pushout property follows from this property for surface stable maps, see [68, Tag 08LQ]. Let $\mathbf{k}(T)$ be the category of morphisms $T \to \mathrm{Spec}(\mathbf{k})$, then the corresponding map

$$\mathbf{k}(T') \to \mathbf{k}(T_2) \times_{\mathbf{k}(T)} \mathbf{k}(T_1)$$

is a bijection, see [68, Lemma 81.6.1.] for the Pushout of Spaces. We have the following diagram:

$$\begin{tikzcd} M^{\mathrm{ind}}(T') \arrow[r] \arrow[d] & M^{\mathrm{ind}}(T_1) \times_{M^{\mathrm{ind}}(T)} M^{\mathrm{tw}}(T_2) \arrow[d] \\ \mathbf{k}(T') \arrow[r] & \mathbf{k}(T_2) \times_{\mathbf{k}(T)} \mathbf{k}(T_1) \end{tikzcd}$$

Then the result again follows from Pushout of Spaces in [68, Lemma 81.6.1.] .

(2) The condition (2) in [10, Theorem 5.3] is the limit preserving property. Let $\hat{A}$ be a complete local algebra over $\mathbf{k}$ with maximal ideal $\mathfrak{m}$, and the residue field is finite type over $\mathbf{k}$. Then the canonical map

$$M^{\mathrm{ind}}(\hat{A}) \to \varprojlim_{l} M^{\mathrm{ind}}(\hat{A}/\mathfrak{m}^l)$$

is an equivalence of groupoids.

Let $\{(\mathfrak{X}^{(l)}, \mathfrak{D}^{(l)})/\hat{A}/\mathfrak{m}^l \xrightarrow{g^{(l)}} \mathfrak{W}\}$ be a formal object on the right hand side, then the Grothendieck Existence Theorem for formal schemes tells us that there exists a formal object

$$(\widehat{\mathfrak{X}}, \widehat{\mathfrak{D}})/\hat{A} \xrightarrow{\hat{g}} \mathfrak{W}$$

on the left hand side. Hence we only need to show that $(\widehat{\mathfrak{X}}, \widehat{\mathfrak{D}})/\hat{A} \xrightarrow{\hat{g}} \mathfrak{W}$ is a stable map over $\hat{A}$. This is from the fact that the semistability condition is a closed condition on the base scheme T and the full faithfulness of the functor by Grothendieck's existence theorem.

(3) The condition (3) in [10, Theorem 5.3] are deformation and obstructions.

Lemma 3.6. *Let A be a $\mathbf{k}$-algebra, and $A \otimes \mathfrak{n}$ the trivial thickening of A. In the scheme level this corresponds to the sheaf of rings* $\mathrm{Spec}(A \otimes \mathfrak{n})$. *Let*

$$((\mathfrak{X}, \mathfrak{D})/\operatorname{Spec} A \to \mathfrak{W}) \in M^{\mathrm{ind}}(\operatorname{Spec} A).$$

Then

(1) *The module of infinitesimal automorphisms is* $T^0_{\mathrm{QG}}((\mathfrak{X},\mathfrak{D})/A \xrightarrow{g} \mathfrak{W}, \mathcal{O}_{\mathfrak{D}}\otimes_A \mathfrak{n})$;

(2) *The module of infinitesimal deformations is* $T^1_{\mathrm{QG}}((\mathfrak{X},\mathfrak{D})/A \xrightarrow{g} \mathfrak{W}, \mathcal{O}_{\mathfrak{D}}\otimes_A \mathfrak{n})$;

(3) *The module* $o((\mathfrak{X},\mathfrak{D})/A \xrightarrow{g} \mathfrak{W}, \overline{A})$ *of obstructions is given by* $T^2_{\mathrm{QG}}((\mathfrak{X},\mathfrak{D})/A \xrightarrow{g} \mathfrak{W}, \mathcal{O}_{\mathfrak{D}}\otimes_A \mathfrak{n})$.

Proof. This is from Theorem 3.4 and standard results in deformation-obstruction theory. □

This lemma verifies Condition (3) and the last part of (1) of [10, Theorem 5.3].

(4) We are left to check Condition (4), which is the "local quasi-separation" property. Let $x := \{(\mathfrak{X},\mathfrak{D})/T \xrightarrow{g} \mathfrak{W}\}$ be an element in $M^{\mathrm{ind}}(T)$ and ϕ an automorphism of x. Suppose that ϕ induces the identity on $M^{\mathrm{ind}}(P)(T)$ for a dense set of points $t\in T$ of finite type, then ϕ is the identity on a dense set of points of finite type on T. Hence ϕ must be the identity on the whole space since $\{(\mathfrak{X},\mathfrak{D})/T \xrightarrow{g} \mathfrak{W}\}$ is flat and separate over T. So from [10, Theorem 5.3], the category M^{ind} is an algebraic stack locally of finite presentation over $\mathbf{k}$.

The moduli functor M^{ind} is bounded, since the coarse moduli space coarsely represents the functor of stable maps from log surface pairs to projective scheme W, which is bounded. Thus, the moduli functor M^{ind} is an algebraic stack of finite type. This finishes the proof. □

Theorem 3.7. *The algebraic stack* $M^{\mathrm{ind}} = \overline{M}^{\mathrm{ind}}_{K^2,I,A,B}(\mathfrak{W})$ *is a projective Deligne-Mumford stack* M^{ind}*. Moreover there exists a proper morphism*

$$\Psi : M^{\mathrm{ind}} \to M = \overline{M}_{K^2,I,A,B}(W)$$

between these two Deligne-Mumford stacks, and this morphism induces a proper map on the coarse moduli spaces.

Proof. To prove M^{ind} is a Deligne-Mumford stack, we show

(1) the diagonal morphism

$$M^{\mathrm{ind}} \to M^{\mathrm{ind}} \times_{\mathbf{k}} M^{\mathrm{ind}}$$

is representable and unramified.

(2) there exists a smooth surjective morphism

$$\mathscr{C} \to M^{\mathrm{ind}}$$

from a scheme $\mathscr{C}$ of finite type over $\mathbf{k}$.

We prove (1). Let us fix two objects

$$((\mathfrak{X},\mathfrak{D})/T \xrightarrow{g} \mathfrak{W})$$

and

$$((\mathfrak{X}',\mathfrak{D}')/T \xrightarrow{g'} \mathfrak{W})$$

in $M^{\mathrm{ind}}(T)$. We show that the isomorphism functor $\mathbf{Isom}_T((\mathfrak{X},\mathfrak{D})/T \xrightarrow{g} \mathfrak{W}, (\mathfrak{X}',\mathfrak{D}')/T \xrightarrow{g'} \mathfrak{W})$ is represented by a quasi-projective group scheme $\mathrm{Isom}_T((\mathfrak{X},\mathfrak{D})/T \xrightarrow{g} \mathfrak{W}, (\mathfrak{X}',\mathfrak{D}')/T \xrightarrow{g'} \mathfrak{W})$ over T. Let $((\mathcal{X},\mathcal{D})/T \xrightarrow{g} W)$ and $((\mathcal{X}',\mathcal{D}')/T \xrightarrow{g'} W)$ be the corresponding $\mathbb{Q}$-Gorenstein families of stable maps to W. Then from Proposition 2.13, the isomorphism functor $\mathbf{Isom}_T((\mathcal{X},\mathcal{D})/T \xrightarrow{g} W,(\mathcal{X}',\mathcal{D}')/T \xrightarrow{g'} W)$ is represented by a quasi-projective group scheme $\mathrm{Isom}_T((\mathcal{X},\mathcal{D})/T \xrightarrow{g} W,(\mathcal{X}',\mathcal{D}')/T \xrightarrow{g'} W)$ over T. Any isomorphism $((\mathfrak{X},\mathfrak{D})/T \xrightarrow{g} \mathfrak{W}) \xrightarrow{\cong} ((\mathfrak{X}',\mathfrak{D}')/T \xrightarrow{g'} \mathfrak{W})$ implies an isomorphism $((\mathcal{X},\mathcal{D})/T \xrightarrow{g} W) \xrightarrow{\cong} ((\mathcal{X}',\mathcal{D}')/T \xrightarrow{g'} W)$. Thus, we get a morphism

$$\mathbb{I}:\mathbf{Isom}_T((\mathfrak{X},\mathfrak{D})/T \xrightarrow{g} \mathfrak{W},(\mathfrak{X}',\mathfrak{D}')/T \xrightarrow{g'} \mathfrak{W}) \to \mathbf{Isom}_T((\mathcal{X},\mathcal{D})/T \xrightarrow{g} W,(\mathcal{X}',\mathcal{D}')/T \xrightarrow{g'} W) \tag{3.3.1}$$

as stacks. Conversely, if we have an isomorphism

$$((\mathcal{X},\mathcal{D})/T \xrightarrow{g} W) \xrightarrow{\cong} ((\mathcal{X}',\mathcal{D}')/T \xrightarrow{g'} W)$$

of stable maps to W, then from Lemma **??**, there are finite liftings of maps to the μ_N-gerbe $\mathfrak{W}$. Thus, if $((\mathfrak{X},\mathfrak{D})/T \xrightarrow{g} \mathfrak{W}) \xrightarrow{\cong} ((\mathfrak{X}',\mathfrak{D}')/T \xrightarrow{g'} \mathfrak{W})$ is one such lifting, then the fiber of $\mathbb{I}$ must be finite. This shows that $\mathbb{I}$ is a finite morphism. Since the right side of (3.3.1) is represented by a quasi-projective group scheme, so does the left side of (3.3.1). It is unramified since the automorphism groups of the geometric fibers are all finite.

We prove (2). Recall from Proposition 2.13, there exists a scheme of finite type

$$\Phi:\mathscr{C}\to M=\overline{M}_{K^2,I,A,B}(W).$$

Here $\mathscr{C}\subseteq \mathrm{Hilb}^h_{\mathbb{P}^M\times\mathbb{P}^{M_1}}$ for large $M>0, M_1>0$ and $\mathrm{Hilb}^h_{\mathbb{P}^M\times\mathbb{P}^{M_1}}$ is the Hilbert scheme of graphs Γ_g of 2-dimensional closed subschemes with Hilbert polynomial h determined by the numerical data K^2, A, B.

Since our s.l.c. pairs $(\mathcal{X},\mathcal{D})$ are all projective, and the Deligne-Mumford stack $(\mathfrak{X},\mathfrak{D})$ is also projective, there exists a weighted projective stack $\mathbb{P}^M_{\mathbf{w}}$ with coarse moduli space $\mathbb{P}^M$ such that

$$(\mathfrak{X},\mathfrak{D})\hookrightarrow \mathbb{P}^M_{\mathbf{w}}.$$

Then for all the maps $g:(\mathfrak{X},\mathfrak{D})\to\mathfrak{W}$, the graphs

$$\Gamma_g\subseteq \mathrm{Hilb}^h_{\mathbb{P}^M_{\mathbf{w}}\times\mathbb{P}^{M_1}(N,\cdots,N)}$$

of the Hilbert stack of maps $(\mathfrak{X},\mathfrak{D}) \xrightarrow{g} \mathfrak{W}$ such that the coarse moduli space of g has Hilbert polynomial h. Let

$$\psi:\mathrm{Hilb}^h_{\mathbb{P}^M_{\mathbf{w}}\times\mathbb{P}^{M_1}(N,\cdots,N)}\to \mathrm{Hilb}^h_{\mathbb{P}^M\times\mathbb{P}^{M_1}}$$

be the map by forgetting the stack structure, and let

$$\widetilde{\mathscr{C}}=\psi^{-1}(\mathscr{C}).$$

Then we have

$$\begin{tikzcd} \widetilde{\mathscr{C}} \arrow[r, "\Phi"] \arrow[dr, "\overline{\Phi}"'] & M^{\mathrm{ind}} \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

where π is the natural morphism from M^{ind} to M by sending $((\mathfrak{X},\mathfrak{D}) \xrightarrow{g} \mathfrak{W}) \mapsto ((\mathcal{X},\mathcal{D}) \xrightarrow{g} W)$. This morphism Φ is constructed from the map $\overline{\Phi}$ to M, and the above diagram. Note that since

M^{ind} may be disconnected, hence $\widetilde{\mathscr{C}}$ may be finite copies of $\mathscr{C}$. Finally Φ is smooth and surjective, since $\overline{\Phi}$ is.

Finally we show that the morphism $\pi : M^{\mathrm{ind}} \to M$ is proper using the valuative criterion for properness. We consider the diagram

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & M^{\mathrm{ind}} \arrow[d, "f^{\mathrm{ind}}"] \\
\mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & M
\end{tikzcd}$$

where R is a valuation ring and K is the field of fractions. Let $\{(\mathcal{X},\mathcal{D})/\operatorname{Spec}(R) \to W\}$ be a family of stable maps from log s.l.c. surface pairs to the projective scheme W. Then from the construction of index one covers there is a unique flat family $\{(\mathfrak{X},\mathfrak{D}) \to \operatorname{Spec}(R)\}$ of index one covering Deligne-Mumford stacks. Now since there exists a lifting

$$g : (\mathfrak{X}_{\mathrm{gen}}, \mathfrak{D}_{\mathrm{gen}}) \to \mathfrak{W}$$

on the generic point of $(\mathfrak{X},\mathfrak{D})$ over $\operatorname{Spec}(K)$, and the above dotted arrow exists and is unique. Grothendieck existence theorem implies that there is a morphism $g : (\mathfrak{X},\mathfrak{D}) \to \mathfrak{W}$, thus the above dotted arrow exists and is unique. Thus, the morphism $\pi : M^{\mathrm{ind}} \to M$ is proper.

It is interesting to describe the fiber of the morphism $\pi : M^{\mathrm{ind}} \to M$. For each geometric point $((X,D) \xrightarrow{g} W) = \operatorname{Spec}(\mathbf{k}) \in M$, we have the log index one covering Deligne-Mumford stack $(\mathfrak{X},\mathfrak{D})$, then the fiber of π is $\operatorname{Hom}^s((\mathfrak{X},\mathfrak{D}),\mathfrak{W})$ which consists of all the stable maps such that its coarse moduli space is $g : (X,D) \to W$. □

Remark 3.8. *In the case that for all the family of log surface pairs $(\mathcal{X},\mathcal{D})$ in the stable map $(\mathfrak{X},\mathfrak{D})/T \xrightarrow{g} \mathfrak{W}$, the log pairs only have l.c.i. singularities, then it is not necessary to take the index one covers. We can just work on M and take $\mathfrak{W} = W$.*

3.4. **Moduli stack of stable maps from** lci **covers.**

Let $\mathfrak{W} \to W$ be a projective Deligne-Mumford stack over $\mathbf{k}$ with projective coarse moduli space W.

First we need to define the family of stable maps from log lci covering Deligen-Mumford stacks to $\mathfrak{W}$. We have the following definition which generalizes [38, Definition 6.17].

Definition 3.9. *We define the families of stable maps from* lci *cover Deligne-Mumford stacks to $\mathfrak{W}$ over a scheme T in the following diagram*

$$\begin{tikzcd}
((\mathfrak{X}^{\mathrm{lci}},\mathfrak{D}^{\mathrm{lci}})/T \to \mathfrak{W}) \arrow[rr, "\hat{\pi}"] \arrow[dr, "\pi^{\mathrm{lci}}"] \arrow[ddr, "f^{\mathrm{lci}}"'] & & ((\mathfrak{X},\mathfrak{D})/T \to \mathfrak{W}) \arrow[dl, "\pi"'] \arrow[ddl, "f"] \\
& ((\mathcal{X},\mathcal{D})/T \to W) \arrow[d, "\overline{f}"] & \\
& T. &
\end{tikzcd} \tag{3.4.1}$$

where

1. *$\overline{f} : ((\mathcal{X},\mathcal{D})/T \to W) \to T$ is a $\mathbb{Q}$-Gorenstein deformation family of stable maps from log s.l.c. surface pairs to W which satisfies the condition in Definition 2.6;*

(2) $f : ((\mathfrak{X}, \mathfrak{D})/T \to \mathfrak{W}) \to T$ *is the corresponding stable maps from the index one covering Deligne-Mumford stack;*
(3) $f^{\mathrm{lci}} : ((\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}})/T \to \mathfrak{W}) \to T$ *is the lifting of the stable map from the log* lci *covering Deligne-Mumford stack, such that the morphism* π^{lci} *factors through the morphism* π*;*
(4) *both* $((\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}})/T \to \mathfrak{W})$ *and* $((\mathfrak{X}, \mathfrak{D})/T \to \mathfrak{W})$ *have the same coarse moduli space* $((\mathcal{X}, \mathcal{D})/T \to W)$*;*
(5) *for the flat family* f^{lci}*, let* (X, D, x) *be a singularity germ in* $X = \overline{f}^{-1}(0)$ *such that* $(\widetilde{X}, x) \to (X, x)$ *is the* lci *cover with transformation group* G*, and the* lci *covering Deligne-Mumford stack* $\mathfrak{X}^{\mathrm{lci}}$ *locally is given by* $[\widetilde{X}/G]$*, we make the following conditions.*
   (a) *suppose that the flat family* $\overline{f} : (\mathcal{X}, \mathcal{D})/T \to W$ *lies on the smoothing component* M^{sm} *(i.e., the component containing stable maps from smooth surface pairs) of* M*. We may assume that* $\overline{f} : \mathcal{X} \to T = \mathrm{Spec}(\mathbf{k}[t])$ *is a one-parameter smoothing of the singularity* (X, D, x)*. If the* lci *cover* $(\widetilde{X}, x)$ *locally is given by*
$$\mathrm{Spec}\, \mathbf{k}[x_1, \cdots, x_\ell]/(h_1, \cdots, h_{\ell-2}),$$
   then the flat family $f^{\mathrm{lci}} : \mathfrak{X}^{\mathrm{lci}} \to T$ *is given by the* G*-equivariant smoothing of the singularity* $(\widetilde{X}, x)$ *which is given by:*
$$\mathrm{Spec}\, \mathbf{k}[x_1, \cdots, x_\ell, t]/(h_1 - t, \cdots, h_{\ell-2} - t),$$
   where G *acts on* t *trivially.*
   (b) *suppose that the flat family* $\overline{f} : ((\mathcal{X}, \mathcal{D})/T \to W) \to T$ *lies on a deformation component of* M *containing the same type of singularities as* (X, D, x)*, then we require that the flat family* $f^{\mathrm{lci}} : ((\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}})/T \to \mathfrak{W}) \to T$ *induces the family* $\overline{f} : ((\mathcal{X}, \mathcal{D})/T \to W) \to T$*.*
(6) *for any singularity germ* (X, D, x) *in a family* $\overline{f} : ((\mathcal{X}, \mathcal{D})/T \to W) \to T$*, if* (X, D, x) *is a simple elliptic singularity, a cusp or a degenerate cusp singularity, or a cyclic quotient of them (these singularities do not pass through* D*), then the* lci *lifting* $(\widetilde{X}, \widetilde{D}, x)$ *is nontrivial such that we have the Deligne-Mumford stack* $[\widetilde{X}/G]$*, we require that they all belong to the* lci *covers constructed in* [38, §6].

We define the moduli functor
$$M^{\mathrm{lci}} := \overline{M}^{\mathrm{lci}}_{K^2, I, A, B}(\mathfrak{W}) : \mathrm{Sch}_{\mathbf{k}} \to \mathrm{Groupoids}$$
which sends for scheme $T \in \mathrm{Sch}_{\mathbf{k}}$,
$$M^{\mathrm{lci}}(T) = \{(\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}})/A \xrightarrow{g} \mathfrak{W}\},$$
the family of stable maps from the log lci surface Deligne-Mumford stack $(\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}})/T$ to $\mathfrak{W}$ in Definition 3.9 such that the inducing maps
$$(\mathcal{X}, \mathcal{D})/A \xrightarrow{g} W$$
on the coarse moduli spaces are stable and satisfy the conditions in Definition 2.6.

Our result for this moduli functor is:

Theorem 3.10. *The moduli functor* $M^{\mathrm{lci}} = \overline{M}^{\mathrm{lci}}_{K^2, I, A, B}(\mathfrak{W})$ *is a projective Deligne-Mumford stack. Moreover there exists a proper morphism*
$$\Psi : M^{\mathrm{lci}} \to M = \overline{M}_{K^2, I, A, B}(W)$$

between these two Deligne-Mumford stacks, and this morphism induces a proper map on the coarse moduli spaces.

Proof. First the moduli functor M^{lci} is represented by an algebraic stack of finite type over $\mathbf{k}$. The proof is the same as Theorem 3.5, and the deformation-obstruction theory of stable maps from log lci covering Deligne-Mumford stacks to $\mathfrak{W}$ is classified in Theorem 3.4.

Next we show M^{lci} is a Deligne-Mumford stack, which is from the following two results:

(1) the diagonal morphism
$$M^{\mathrm{lci}} \to M^{\mathrm{lci}} \times_{\mathbf{k}} M^{\mathrm{lci}}$$
is representable and unramified.

(2) there exists a smooth surjective morphism
$$\mathscr{C}^{\mathrm{lci}} \to M^{\mathrm{lci}}$$
from a scheme $\mathscr{C}^{\mathrm{lci}}$ of finite type over $\mathbf{k}$.

For (1), let $((\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}})/T \xrightarrow{g} \mathfrak{W})$ and $((\mathfrak{X}'^{\mathrm{lci}}, \mathfrak{D}'^{\mathrm{lci}})/T \xrightarrow{g'} \mathfrak{W})$ be two objects in $M^{\mathrm{lci}}(T)$. We need to show that the isomorphism functor
$$\mathbf{Isom}(\mathfrak{X}^{\mathrm{lci}}, \mathfrak{X}'^{\mathrm{lci}}) := \mathbf{Isom}_T(((\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}})/T \xrightarrow{g} \mathfrak{W}), ((\mathfrak{X}'^{\mathrm{lci}}, \mathfrak{D}'^{\mathrm{lci}})/T \xrightarrow{g'} \mathfrak{W})$$
is represented by a quasi-projective group scheme
$$\mathrm{Isom}_T(((\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}})/T \xrightarrow{g} \mathfrak{W}), ((\mathfrak{X}'^{\mathrm{lci}}, \mathfrak{D}'^{\mathrm{lci}})/T \xrightarrow{g'} \mathfrak{W})$$
over T. Let $((\mathcal{X}, \mathcal{D})/T \xrightarrow{g} W)$ and $((\mathcal{X}', \mathcal{D}')/T \xrightarrow{g'} W)$ be the corresponding $\mathbb{Q}$-Gorenstein families of stable maps to W. Proposition 2.13 implies that the isomorphism functor $\mathbf{Isom}(\mathcal{X}, \mathcal{X}') := \mathbf{Isom}_T((\mathcal{X}, \mathcal{D})/T \xrightarrow{g} W, (\mathcal{X}', \mathcal{D}')/T \xrightarrow{g'} W)$ is represented by a quasi-projective group scheme over T. Consider the following diagram
$$\begin{tikzcd}
(\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}}) \arrow[r, "\cong"] \arrow[d, "\pi"'] & (\mathfrak{X}'^{\mathrm{lci}}, \mathfrak{D}'^{\mathrm{lci}}) \arrow[d, "\pi"] \\
(\mathcal{X}, \mathcal{D}) \arrow[r, "\cong"] & (\mathcal{X}', \mathcal{D}'),
\end{tikzcd}$$
where the vertical arrows are maps to their coarse moduli spaces. Any isomorphism $((\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}})/T \xrightarrow{g} \mathfrak{W}) \xrightarrow{\cong} ((\mathfrak{X}'^{\mathrm{lci}}, \mathfrak{D}'^{\mathrm{lci}})/T \xrightarrow{g'} \mathfrak{W})$ induces an isomorphism $((\mathcal{X}, \mathcal{D})/T \xrightarrow{g} W) \xrightarrow{\cong} ((\mathcal{X}', \mathcal{D}')/T \xrightarrow{g'} W)$. Thus, we get a morphism of functors
$$\mathbb{I} : \mathbf{Isom}(\mathfrak{X}^{\mathrm{lci}}, \mathfrak{X}'^{\mathrm{lci}}) \to \mathbf{Isom}(\mathcal{X}, \mathcal{X}'). \tag{3.4.2}$$

Conversely, if we have an isomorphism $((\mathcal{X}, \mathcal{D})/T \xrightarrow{g} W) \xrightarrow{\cong} ((\mathcal{X}', \mathcal{D}')/T \xrightarrow{g'} W)$ of stable maps to W, then if there is a lifting
$$((\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}})/T \xrightarrow{g} \mathfrak{W}) \cong ((\mathfrak{X}'^{\mathrm{lci}}, \mathfrak{D}'^{\mathrm{lci}})/T \xrightarrow{g'} \mathfrak{W}),$$

the fiber of $\mathbb{I}$ must be finite. This shows that $\mathbb{I}$ is a finite morphism. Since the right side of (3.4.2) is represented by a quasi-projective group scheme, so does the left side of (3.4.2). It is unramified since the automorphism groups of the geometric fibers are all finite.

For (2), from Proposition 2.13, there exists a finite type scheme $\Phi : \mathscr{C} \to M = \overline{M}_{K^2,I,A,B}(W)$. Here $\mathscr{C} \subseteq \mathrm{Hilb}^h_{\mathbb{P}^M \times \mathbb{P}^{M_1}}$ for large $M > 0, M_1 > 0$ and $\mathrm{Hilb}^h_{\mathbb{P}^M \times \mathbb{P}^{M_1}}$ is the Hilbert scheme of graphs Γ_g of 2-dimensional closed subschemes with Hilbert polynomial h determined by the numerical data K^2, A, B.

The proof below is similar to Theorem 3.7. Our s.l.c. pairs $(\mathcal{X}, \mathcal{D})$ and the log Deligne-Mumford stack $(\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}})$ are all projective, then there exists a weighted projective stack $\mathbb{P}^M_{\mathbf{w}}$ with coarse moduli space $\mathbb{P}^M$ such that

$$(\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}}) \hookrightarrow \mathbb{P}^M_{\mathbf{w}}.$$

Then for all the maps $g : (\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}}) \to \mathfrak{W}$, the graphs

$$\Gamma_g \subseteq \mathrm{Hilb}^h_{\mathbb{P}^M_{\mathbf{w}} \times \mathbb{P}^{M_1}(N,\cdots,N)}$$

of the Hilbert stack of maps $(\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}}) \xrightarrow{g} \mathfrak{W}$ such that the coarse moduli space of g has Hilbert polynomial h. Let

$$\psi : \mathrm{Hilb}^h_{\mathbb{P}^M_{\mathbf{w}} \times \mathbb{P}^{M_1}(N,\cdots,N)} \to \mathrm{Hilb}^h_{\mathbb{P}^M \times \mathbb{P}^{M_1}}$$

be the map by forgetting the stacky structure, and let

$$\mathscr{C}^{\mathrm{lci}} = \psi^{-1}(\mathscr{C}).$$

Then we have

$$\begin{tikzcd} \mathscr{C}^{\mathrm{lci}} \arrow[r, "\Phi"] \arrow[dr, "\overline{\Phi}"'] & M^{\mathrm{lci}} \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

where π is the natural morphism from M^{lci} to M by sending $((\mathfrak{X}^{\mathrm{lci}}, \mathfrak{D}^{\mathrm{lci}}) \xrightarrow{g} \mathfrak{W}) \mapsto ((\mathcal{X}, \mathcal{D}) \xrightarrow{g} W)$. This morphism Φ is constructed from the map $\overline{\Phi}$ to M, and the above diagram. Note that since M^{lci} maybe disconnected, hence $\mathscr{C}^{\mathrm{lci}}$ maybe finite copies of $\mathscr{C}$. Finally Φ is smooth and surjective, since $\overline{\Phi}$ is.

The final step is to show that the morphism $\pi : M^{\mathrm{lci}} \to M$ is proper. We generalize [38, Theorem 6.19] to the case of stable maps. We use the valuative criterion for properness and consider the following diagram

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] \arrow[d] & M^{\mathrm{lci}} \arrow[d, "f^{\mathrm{lci}}"] \\ \mathrm{Spec}(R') \arrow[r] \arrow[urr, dashed] & \mathrm{Spec}(R) \arrow[r] \arrow[ur, dashed] & M \end{tikzcd}$$

where R is a valuation ring with field of fractions K, and residue field $\mathbf{k}$. We can take $R = \mathbf{k}[\![t]\!]$ and $K = \mathbf{k}(\!(t)\!)$. The morphism $\mathrm{Spec}(R) \to M$ corresponds a flat $\mathbb{Q}$-Gorenstein family $(\mathcal{X}, \mathcal{D}) / \mathrm{Spec}(R) \xrightarrow{g} W$ of stable maps from log s.l.c. surface pairs to W. We may assume that $\mathrm{Spec}(R) \to M$ lies on the smoothing component of the moduli stack M, since if $\mathrm{Spec}(R) \to M$ lies in other component of M, then from condition (6) in Definition 3.9 we alway have that the family $(\mathcal{X},\mathcal{D})/\operatorname{Spec}(R)\stackrel{g}{\to}W$ is induced from a flat family of lci covering DM stacks.

Thus, we only need to consider the singularity germs $(\mathcal{X},\mathcal{D},x)$ in $\mathcal{X}$ for the family of stable maps $(\mathcal{X},\mathcal{D})/\operatorname{Spec}(R)\stackrel{g}{\to}W$. Then we use the same proof as in [38, Theorem 6.19] to show that after a finite morphism $\operatorname{Spec}(R')\to\operatorname{Spec}(R)$, the morphism $\operatorname{Spec}(K')\to M^{\mathrm{lci}}$ can be uniquely lifted to a morphism $\operatorname{Spec}(R')\to M^{\mathrm{lci}}$ which completes the valuative criterion for properness. □

4. Moduli space of Hassett stable pairs

Our first example of $M=\overline{M}_{K^2,A,B,I}(W)$ is the moduli space of log surface pairs when $W=pt$ is a point.

Let $C_d\subset\mathbb{P}^2$ be a smooth plane curve of degree $d\geq 4$. Then $(\mathbb{P}^2,C_d)$ is a stable log surface pair. Let $\mathbb{P}(\operatorname{Sym}^d(\mathbb{C}^3))$ be the linear system of degree d plane curves and $U\subset\mathbb{P}(\operatorname{Sym}^d(\mathbb{C}^3))$ the open subset parametrizing smooth degree d curves. The GIT quotient $\mathcal{P}_d=U/\operatorname{PGL}_3$ is a quasi-projective variety. From [33], let

$$f:\mathcal{P}_d\to M_{g(d)}$$

be the morphism to the moduli space of genus $g(d)=\frac{1}{2}(d-1)(d-2)$ curves. Let $\mathcal{M}^{\mathrm{sm}}\subset\overline{\mathrm{KSBA}}$ be the smoothing component of stable log surface pairs in the moduli space $\overline{\mathrm{KSBA}}$ parametrizing log surface pairs (X,D) with the same topological invariants as $(\mathbb{P}^2,C_d)$. Let $\overline{\mathcal{P}}_d\subset\mathcal{M}$ be the closure of $\mathcal{P}_d$ in $\mathcal{M}$ and

$$f:\overline{\mathcal{P}}_d\to\overline{M}_{g(d)}$$

be the forgetting morphism extending the morphism $f:\mathcal{P}_d\to M_{g(d)}$. Such an extension exists since for any $(X,D)\in\mathcal{M}\subset\overline{\mathrm{KSBA}}$, D is nodal. Hacking [29] studied the general case $d\geq 4$ for $\mathcal{M}$. Let us only fix to $d=4$. We have

Theorem 4.1. ([33, Theorem 1]) *The morphism $f:\overline{\mathcal{P}}_4\to\overline{M}_3$ is an isomorphism.*

First from [33, Corollary 2.4], if C is a stable curve of genus three and also 3-connected and not contained in the closure of the hyperelliptic locus, then C is canonically embedded as a plane curve inside $\mathbb{P}^2$ such that $(\mathbb{P}^2,C)$ is stable. Thus the morphism

$$f:\overline{\mathcal{P}}_4\to\overline{M}_3$$

is definitely isomorphic over the locus (X,D) in $\overline{\mathcal{P}}_4$ such that the curves D are not 1-connected, 2-connected, and 3-connected hyperelliptic curves.

4.1. **Hyperelliptic curves.** Over a smooth hyperelliptic curve C which gives a double cover $\pi: C\to\mathbb{P}^1$, from [33, §4.1],

$$\pi_*(\mathcal{O}_C)=\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(4).$$

Then C is embedded into the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(4))$ as a bisection and is disjoint from the zero section $\mathbb{P}^1$ with $(\mathbb{P}^1)^2=-4$. Blowing down this zero section $\mathbb{P}^1$, and we get the weighted projective space $\mathbb{P}(1,1,4)$. The pair $(\mathbb{P}(1,1,4),C)$ is stable. Since $\mathbb{P}(1,1,4)$ can be deformed to $\mathbb{P}^2$ as a Veronese surface, the smooth hyperelliptic curve C is deformed to singular hyperelliptic curves in $\mathbb{P}^2$. The morphism $f : \overline{\mathcal{P}}_4 \to \overline{M}_3$ is defined as:

$$(\mathbb{P}(1,1,4), C) \mapsto C$$

over such a hyperelliptic curve. From the deformation theory arguments of the stable pairs, f is injective over these hyperelliptic locus.

4.2. **2-connected curves.** Let C be a stable curve which is 2-connected but not three connected, then

$$C = C_1 \cup_{p,q} C_2$$

where C_1, C_2 are curves of arithmetic genus one. Each C_i is either a smooth elliptic curve, an irreducible nodal curve of arithmetic genus one, or the union of two rational curves meeting at two points. [33, §4.2] showed that such a curve C is embedded into

$$C \hookrightarrow \mathbb{P}(1,1,2) \cup \mathbb{P}(1,1,2).$$

The construction of $\mathbb{P}(1,1,2) \cup \mathbb{P}(1,1,2)$ is given as follows. From [33, §4.2], $C = C_1 \cup C_2$ is naturally embedded into the union of two ruled surfaces $\mathbb{F}_2$ glued along a ruling B. The curve C is disjoint from the zero section of $\mathbb{F}_2 \cup \mathbb{F}_2$. Thus we blow down the zero section and get the surface $X = \mathbb{P}(1,1,2) \cup \mathbb{P}(1,1,2)$ and the stable pair (X, C). This surface $X = \mathbb{P}(1,1,2) \cup \mathbb{P}(1,1,2) \hookrightarrow \mathbb{P}^5$ can be embedded into $\mathbb{P}^5$ as the cone over the union of two conic curves meeting at a single point. Also from the deformation theory of stable pairs [33, §4.2] proved that the morphism f is an isomorphism over such locus.

4.3. **1-connected curves.** There are several cases of 1-connected curves. The generic 1-connected curve is $C = C_1 \cup_p C_2$, where C_1 and C_2 are irreducible genus two and genus one curves respectively, and $p \in C_1$ is not Weierstrass. In this case we draw the following diagram

$$\begin{tikzcd} C_1 \arrow[r, hook] \arrow[d] & \mathbb{F}_2 \arrow[d, "\text{toriodal blow-up}"] \\ C^{\text{cusp}} \arrow[r, hook] & \mathbb{P}^2 \end{tikzcd}$$

where C^{cusp} is a cuspidal quartic in $\mathbb{P}^2$, and $\mathbb{F}_2 \to \mathbb{P}^2$ is the toriodal blow-up. We form the surface $X = \mathbb{F}_2 \cup_B \mathbb{P}(1,2,3)$ which is the glueing of $\mathbb{F}_2$ with $\mathbb{P}(1,2,3)$ along the ruling B. The curve $C_2 \subset \mathbb{P}(1,2,3)$ is a nodal curve generating the Picard group of $\mathbb{P}(1,2,3)$. [33, §4.4] proved that the log surface pair (X, C) is stable. Also the morphism f is an isomorphism over such locus, and every deformation of X forces C to deform to a 3-connected curve.

Other cases are the degenerate cases of the generic 1-connected curve. [33, Lemma 4.2] listed all the cases of pairs (P, C_0) that the surface X can be obtained from. Here $C_0 \subset P$ is a hypersurface quadric section with only nodes and cusps as singularities.

(1) $P = \mathbb{P}^2$, C_0 has one single cusp.
(2) $P = \mathbb{P}(1,1,4)$, C_0 has one cusp.
(3) $P = \mathbb{P}(1,1,2) \cup \mathbb{P}(1,1,2)$, C_0 has one cusp.
(4) $P = \mathbb{P}^2$, C_0 has two cusps.
(5) $P = \mathbb{P}(1,1,4)$, C_0 has two cusps.
(6) $P = \mathbb{P}(1,1,2) \cup \mathbb{P}(1,1,2)$, C_0 has one cusp on each component.
(7) $P = \mathbb{P}^2$, C_0 has 3 cusps as in [33, §4.5.1].

Here a cusp $p \in C_0$ means that locally it is given by the local analytic equation

$$y^2 = x^3, \text{ in } \operatorname{Spec} \mathbb{C}[x, y].$$

Then we perform the toroidal blow-ups

$$b : S_1 \to P$$

on each cusp singularity and then glue a number of (depending on the number of cusps) $\mathbb{P}(1,2,3)$ along the rulings B_i as in [33, Proposition 4.1]. We denote by $\{B_i\}$ the set of rulings in $\mathbb{P}(1,2,3)$. Then we have

$$X = S_1 \cup_{\{B_i\}} \mathbb{P}(1,2,3)$$

and

$$C = C_1 \cup_{\{p_i\}} C_{2,i}$$

where each $C_{2,i} \subset \mathbb{P}(1,2,3)$ is the nodal curve in $\mathbb{P}(1,2,3)$ generating the Picard group of $\mathbb{P}(1,2,3)$. Then the log surface pair (X, C) is stable and this gives all the degenerate cases of the 1-connected curves. The morphism $f : \overline{\mathcal{P}}_4 \to \overline{M}_3$ is an isomorphism over all the locus of 1-connected curves.

4.4. **Moduli space of index one covers.** From the analysis above in §4.1, §4.2 and §4.3, the singularities of any log surface pair (X, D) in the moduli space M are all klt singularities, which are quotient singularities. Thus, the index one cover of such singularities are all A_l-type singularities which are locally complete intersection singularities. Recall that the moduli stack $M^{\mathrm{ind}} := \overline{M}^{\mathrm{ind}}_{K^2,\chi}$ of index one covers in [38, Theorem 1.1], the moduli stack M^{ind} is isomorphic to the moduli stack M of log surface pairs.

5. Moduli space of stabler maps from Hacking pairs

Let $W = \mathbb{P}^r$ be the projective space of dimension $r > 1$. For a log surface pair (X, D) and a stable map $f : (X, D) \to \mathbb{P}^r$, let us fix the invariants

$$K^2 = (K_X + D)^2 = 1, \quad d = (K_X + D) \cdot H, \quad H^2 = \frac{d}{4}, \quad I = \{1\}.$$

We are interested in the moduli space $\overline{M}_{K^2=1,A=d,B=\frac{d}{4},I}(\mathbb{P}^r)$ of stable maps from log surface pairs (X, D) to $\mathbb{P}^r$ with the given topological invariants. We let

$$\overline{M}^{\mathrm{sm}}_d := \overline{M}^{\mathrm{sm}}_{K^2=1,d,\frac{d}{4},I}(\mathbb{P}^r)$$

be the smoothing component of the moduli stack of stable map spaces. For any stable map $f : (X, D) \to \mathbb{P}^r$, we have the following diagram

$$\begin{tikzcd} D \arrow[r, "\overline{f}"] \arrow[d, hook] & \mathbb{P}^r \arrow[d, "\mathrm{id}"] \\ (X, D) \arrow[r, "f"] & \mathbb{P}^r. \end{tikzcd}$$

Fromm [3, Lemma 2.3], for the stable map $f:(X,D)\to \mathbb{P}^r$, K_X+D+4H is ample, and K_X+D+3H is nef. Thus, we have

$$\begin{aligned}(K_X+D+3H)|_D &= (K_X+D)|_D\otimes 3H|_D\\ &= K_D\otimes 3\overline{f}^*\mathcal{O}_{\mathbb{P}^r}(1).\end{aligned}$$

Over the curve D, $K_D\otimes 3\overline{f}^*\mathcal{O}_{\mathbb{P}^r}(1)$ is nef since K_X+D+3H is nef.

Over the locus of stable maps

$$f:(X,D)\to \mathbb{P}^r$$

such that K_X+D+3H is big, the restriction $K_D\otimes 3\overline{f}^*\mathcal{O}_{\mathbb{P}^r}(1)$ is positive, hence is ample. Thus we get a rational morphism

$$\Phi:\overline{M}_d^{\mathrm{sm}} \dashrightarrow \overline{M}_3(\mathbb{P}^r,d) \tag{5.0.1}$$

It is interesting to study whether the above morphism can be extended to a real morphism.

5.1. **Hacking stable pairs.** Let us first recall the definition of stable pairs in [29].

Definition 5.1. *A semi-log-canonical* (slc) *or semi-log-terminal* (slt) *log surface pair is a pair* (X,D)*, where* X *is a projective surface and* D *is an effective* $\mathbb{Q}$*-divisor such that the following conditions are satisfied:*

(1) X *is Cohen-Macaulay and has only normal crossing singularities in codimension one.*
(2) the divisor K_X+D *is* $\mathbb{Q}$*-Cartier.*
(3) let $X^\nu\to X$ *be the normalization of* X *and let* $\Delta\subset X$ *be the double curve. Let* D^ν,Δ^ν *be the inverse images of* D,Δ *in* X^ν*. Then* $(X^\nu,\Delta^\nu+D^\nu)$ *is log canonical (rep. log terminal).*

Remark 5.2. *The* slc *condition implies that no component of* D *is contained in the double curve* Δ *by* (3).

Definition 5.3. *Let* X *be a projective surface and* D *is an effective* $\mathbb{Q}$*-Cartier divisor. Let* $d\geq 3$ *be an integer. The pair* (X,D) *is stable of degree* d *if the following conditions are satisfied:*

(1) the pair $(X,(\frac{3}{d}+\epsilon)D)$ *is* slc *and the divisor* $K_X+(\frac{3}{d}+\epsilon)D$ *is ample for some* $\epsilon>0$.
(2) the divisor dK_X+3D *is linearly equivalent to zero.*
(3) there is a deformation $(\mathcal{X},\mathcal{D})/T$ *of the pair* (X,D) *over the germ of a curve such that the general fiber* $\mathcal{X}_t$ *of* $\mathcal{X}/T$ *is isomorphic to* $\mathbb{P}^2$ *and the divisors* $K_{\mathcal{X}}$ *and* $\mathcal{D}$ *are* $\mathbb{Q}$*-Cartier.*

From [29, Proposition 6.1, Proposition 6.2], the slt singularities of X are of the following types:

(1) $\frac{1}{n^2}(1,na-1)$, where $(a,n)=1$ and $3\nmid n$.
(2) $(xy=0)\subset \frac{1}{r}(1,-1,a)$, where $(a,r)=1$.
(3) $(x^2=zy^2)\subset \mathbb{A}^3$.

The index of X equals to $n,r,1$ in the above cases respectively. [29, Theorem 4.5] proved that the index $n\leq d$ and $r\leq d$. For all the other strictly slc singularities of X, D must miss such singular points, and $dK_X+3D\sim 0$. Thus, the index of X at these points are at most d.

5.2. **Stable maps from Hacking stable pairs.** Let $\{(\mathcal{X},\mathcal{D})/T \xrightarrow{f} \mathbb{P}^r\}$ be a flat family of stable maps from pairs (X,D) to $\mathbb{P}^r$ with topological invariants (K^2,A,B,I). Then we have a flat family

$$(\mathcal{X},\mathcal{D}=4H)/T$$

of stable pairs in Definition 5.3.

Proposition 5.4. *(*[29, Theorem 7.1]*) Let* $\{(X,D)/T \xrightarrow{f} \mathbb{P}^r\}$ *be a stable map with topological invariants* (K^2,A,B,I) *such that* $\deg(D+4H)$ *is not a multple of* 3*. Then* X *is* slt *and* X *is either a normal log terminal surface or a surface of type B in* [29]*, i.e.,* X *has two normal components meeting in a smooth rational curve.*

Proof. If $\{(X,D)/T \xrightarrow{f} \mathbb{P}^r\}$ is a stable map, then K_X+D is f-ample which means that K_X+D+4H is ample. We take $D+4H$ as an effective $\mathbb{Q}$-Cartier divisor. Since $3 \not| \deg(D+4H)$, then applying [29, Theorem 7.1] we get the result. □

Now we fix the invariants:

$$K^2=1, \quad H\cdot D=d, \quad H^2=\frac{d^2}{16}.$$

Let C_4 be a plane curve of degree 4 in $\mathbb{P}^2$. The log surface pair $(\mathbb{P}^2,C_4)$ has invariants $(K_{\mathbb{P}^2}+C_4)^2=1$, $H\cdot C_4=d$ and $H^2=\mathcal{O}_{\mathbb{P}^2}(\frac{d}{4})^2=\frac{d^2}{16}$.

Theorem 5.5. *Let* $\overline{M}^{sm} := \overline{M}^{sm}_{1,d,\frac{d^2}{16},\{1\}}(\mathbb{P}^r)$ *be the smoothing component of the moduli space of stable maps from log surface pairs* (X,D) *to* $\mathbb{P}^r$ *with topological invariants* $(1,d,\frac{d^2}{16},I=\{1\})$*. If* $d=4$*, then all the possible pairs* (X,D) *in the smoothing component are the following:*

(1) $X=\mathbb{P}^2, \quad \mathbb{P}(1,1,4), \quad \mathbb{P}(1,1,2)\cup\mathbb{P}(1,1,2)$.
(2) D *has following singularities: the cusp singularity* $y^2+x^3=0$*, the singularity* $(xy=0)\subset\frac{1}{2}(1,1,1)$ *in* $\mathbb{P}(1,1,4), \quad \mathbb{P}(1,1,2)\cup\mathbb{P}(1,1,2)$*. Finally* D *must avoid the singular point* $\frac{1}{4}(1,1)$*.*

Proof. Let $(X,D) \xrightarrow{f} \mathbb{P}^r$ be a stable map in $\overline{M}^{\mathrm{sm}}$, then $(X,D+4H)$ is relatively f-ample and it is a stable pair in Definition 5.3 of degree $4+d$. We already assume that $3 \not| 4+d$. Thus, the pair $(X,D+4H)$ must satisfy the condition in [29, Lemma 9.5, Theorem 4.5]. Thus, from [29, Proposition 11.2], the classifications of X and D's can be listed as in the cases of degree 4 and 5 cases in [29, §11.1, §11.2]. For our purpose, since X is either a normal log terminal surface, or a type B surface which both have $K_X^2=9$, we perform the same method in [29, §11.1, §11.2] and try all the possible weighted projective surfaces $\mathbb{P}(a,b,c)$ with $a^2+b^2+c^2=3abc$; and all possible unions of two copies of $\mathbb{P}(a,b,c)$'s with $a,b,c\le d$. The weighted projective surfaces are the only surfaces smoothable to $\mathbb{P}^2$, which are called Manetti surfaces. Now we calculate the result for $d=4$. By calculation only the following cases satisfy $K_X^2=9$:

$$X=\mathbb{P}^2, \quad \mathbb{P}(1,1,4), \quad \mathbb{P}(1,1,2)\cup\mathbb{P}(1,1,2), \quad \mathbb{P}(1,4,25);$$

$$X_{26}\subset\mathbb{P}(1,2,13,25), \quad \mathbb{P}(1,1,5)\cup X_6(\subset\mathbb{P}(1,2,3,5)), \quad \mathbb{P}(1,1,5)\cup\mathbb{P}(1,4,5),$$

where $X_{26}\subset\mathbb{P}(1,2,13,25)$ is the surface obtained from $\mathbb{P}(1,4,25)$by smoothing the singularity $\frac{1}{4}(1,1)$. The smoothing was realised inside $\mathbb{P}(1,2,13,25)$, see [29, §11.2]. But we require that $K^2=$

$(K_X+D)^2=1$, thus the only cases are $X=\mathbb{P}^2, \quad \mathbb{P}(1,1,4), \quad \mathbb{P}(1,1,2)\cup\mathbb{P}(1,1,2)$. Then the divisor D may only contain nodal and cusp singularities $(y^2=x^3)$ in [33] and [29, §11.1]. $\square$

5.3. **The rational morphism.** Recall the rational morphism

$$\Phi : \overline{M}_4^{\mathrm{sm}} \dashrightarrow \overline{M}_3(\mathbb{P}^r, d) \tag{5.3.1}$$

in (5.0.1) when $d=4$. We denote by $\overline{M}_3^{\mathrm{sm}}(\mathbb{P}^r,d)\subset\overline{M}_3(\mathbb{P}^r,d)$ the smoothing component (i.e., the main component containing the stable maps from smooth genus three curves).

Proposition 5.6. *There exists a birational morphism $\Phi : \overline{M}_4^{sm} \dashrightarrow \overline{M}_3^{sm}(\mathbb{P}^r,d)$.*

Proof. Recall that $\overline{M}_4^{\mathrm{sm}} := \overline{M}^{\mathrm{sm}}_{K^2=1,4,1,I}(\mathbb{P}^r)$ and $I=\{1\}$. So if there is a stable map $f:(\mathbb{P}^2,C_4)\to\mathbb{P}^r$ where the C_4 is a smooth degree 4 hypersurface, it definitely induces a stabel map $C_4\to\mathbb{P}^r$. The locus containing $(\mathbb{P}^2,C_4)$ and C_4 are open inside the moduli spaces $\overline{M}_4^{\mathrm{sm}}$ and $\overline{M}_3^{\mathrm{sm}}(\mathbb{P}^r,d)$ respectively. Thus, we get a birational morphism between the moduli sapces. $\square$

Remark 5.7. *In the case that the stable maps have degree zero, the moduli space is just the moduli space of log stable pairs. The papers* [1] *and* [2] *studied the wall crossing of moduli spaces by varying the coefficients I of the divisor D in (X,D) when $\mathbb{P}^2$ deforms to X. It is more difficult to figure the wall crossing in the case of higher degree stable maps.*

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